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Integrable systems on b-symplectic manifolds

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Chapter 1

Introduction

Symplectic geometry originated as a language to describe Hamiltonian systems in classical mechanics and is used as a tool in other areas of physics such as geometrical optics and thermodynamics [AG]. The classical equations of motion of a Hamiltonian system with Hamiltonian function H can be understood as the flow of the Hamiltonian vector field X_H with respect to a certain symplectic form.

For the reader unfamiliar with Hamiltonian systems, we want to illustrate the concept by looking at the equations of motion of a particle of unit mass moving in $\mathbb{R}^3$ under the influence of a central force field with potential V. Newton's law of motion implies that the position x of the particle is given by the second-order differential equation

$$\ddot{x} = F(x) := -\frac{\partial V}{\partial x}, \tag{1.1}$$

where $F(x)$ is the force of the field at the point x. Let $p = \dot{x}$ be the momentum of the particle. The total energy is the sum of kinetic and potential energy,

$$H(x,p) = E_{kin} + V = \frac{|p|^2}{2} + V(x).$$

The manifold of possible positions of the particle is called *configuration space*, in this case $\mathbb{R}^3$, the manifold of positions and momenta is *phase space*, in this case $\mathbb{R}^3 \times \mathbb{R}^3$. Instead of the second-order differential equation (1.1), we can consider the system of first-order differential equations on phase space:

$$\dot{x} = p, \quad \dot{p} = F(x).$$

In the present easy example this step is trivial. However, we can formulate this system of equations as the flow of a vector field on the phase space, the *Hamiltonian vector field* of H. Consider the non-degenerate two-form $\omega_0 = \sum_i dx_i \wedge dp_i$ and define the vector field X_H via $\iota_{X_H}\omega = -dH$. Then we see that X_H is given by $p\frac{\partial}{\partial x} + F(x)\frac{\partial}{\partial y}$, precisely the vector field corresponding to the above system of equations.

In general, the phase space is a more complicated manifold M and instead of ω_0 we consider any *symplectic form* on M, that is, a closed non-degenerate two-form[1]. A Hamiltonian system is then given by a triple (M, ω, H) of a manifold M endowed with a symplectic form ω and a smooth function $H : M \to \mathbb{R}$.

We can go one step further and consider a generalization of symplectic manifolds, so-called Poisson manifolds. Performing a non-symplectic transformation of a Hamiltonian system may yield such a structure; we will see examples later on in this thesis (Chapter 8). A Poisson structure Π is defined as a section of $TM \wedge TM$ such that the induced bracket on functions $\{f, g\} := \Pi(df, dg)$ satisfies the Jacobi identity. A symplectic structure can be viewed as a Poisson structure by using the induced isomorphism between TM and T^*M. However, Poisson structures are much more general than symplectic structures; indeed they induce a (possibly singular) foliation of the manifold into *symplectic leaves*.

The manifolds that will play the central role in this thesis are b-Poisson manifolds. These manifolds are a special sub-class of Poisson manifolds which are in many ways close to being symplectic: For Π the Poisson structure dual to a symplectic form, the top wedge Π^n never meets the zero section of $\Lambda^{2n}TM$ (non-degeneracy of the symplectic form). We define a b-Poisson manifold to be a Poisson manifold (M, Π) such that Π^n vanishes *transversally* to the zero section in $\Lambda^{2n}TM$. In particular, the vanishing set is a hypersurface Z, called the *critical hypersurface*. Moreover, a b-Poisson structure can be dualized to obtain a two-form which is symplectic away from Z and has a controlled singularity on Z. The precise way to formalize this singular behaviour is the concept of b-forms, which give rise to a differential complex called the *b-de Rham complex*, in perfect analogy to the smooth case. A b-Poisson structure then has a dual b-two-form, which is closed and non-degenerate. We therefore call it a **b-symplectic form**. The existence of a "b-Darboux" theorem is another parallel to the symplectic world: it states

[1] See [Ca] for an introduction to symplectic manifolds.

that locally around a point m in Z, a b-symplectic form can be written as

$$\frac{1}{t}\,dz \wedge dt + \sum_{i=1}^{n-1} dx_i \wedge dy_i$$

where $(z, t, x_1, y_1, \ldots, x_{n-1}, y_{n-1})$ are coordinates centered at m and t is a local defining function for Z. The geometry of b-symplectic manifolds has become an active field of research and has been extensively studied in [GMP11, GMP14, GMPS13, FMM, GL, GLPR, MO13].

Coming back to classical Hamiltonian systems, a particularly interesting case is that of **integrable Hamiltonian systems**. Such a system on a $2n$-dimensional symplectic manifold (M, ω) consists of n functions $f_1, \ldots, f_n$ which are independent and commute with respect to the Poisson structure induced by ω, i.e. $\omega(X_{f_i}, X_{f_j}) = 0$. The Hamiltonian function H of the system is assumed to be one of the integrals. A pivotal result about the dynamics of integrable systems is the Liouville-Arnold-Mineur theorem (or *action-angle coordinate theorem*), Theorem 3.1.2, which states that the compact common level sets of the integrals f_i are tori that are invariant under the motion of the system and on which the motion is linear. The action-angle coordinates can be computed explicitly by integration [L, K, AG]; we say that the problem is solvable by *quadratures*, that is to say, using only algebraic operations, differentiation and integration of functions.

The main goal of this thesis is to explore integrable systems in the b-setting, including the proof of an action-angle coordinate theorem and a KAM theorem[2], which is a statement about the behaviour of such systems under certain perturbations. We give a more detailed outline of the results below.

1.1 Structure and results of this thesis

1.1.1 Chapter 2: Preliminaries

We review important results about Poisson and b-symplectic manifolds and consider other types of "singular" symplectic manifolds, namely b^m-symplectic and m-folded symplectic manifolds. Moreover, we define Hamiltonian torus actions on b-symplectic manifolds and the concept of cotangent lifts. We end the chapter with an introduction to the classical KAM theory.

[2] The letters KAM come from the initials of Kolmogorov, Arnold and Moser who initiated the theory in the classical symplectic case.

1.1.2 Chapter 3: Integrable systems on symplectic and Poisson manifolds

A separate chapter is devoted to introducing integrable systems in a variety of settings: Poisson, b-symplectic, commutative, non-commutative. We present the action-angle coordinate theorem in the respective cases.

1.1.3 Chapter 4: Action-angle coordinates for b-integrable systems

In this chapter we prove the existence of action-angle coordinates for b-integrable systems. This result was published in [KMS] (joint work with Eva Miranda and Geoffrey Scott).

Theorem A (Action-angle coordinates for b-integrable systems)**.** *Let (M, ω) be a b-symplectic manifold with critical hypersurface Z. Let F be a b-integrable system on (M, ω) and let $m \in Z$ be a regular point of the system lying inside the critical hypersurface. Assume that the integral manifold $\mathcal{F}_m$ containing m is compact, i.e. a Liouville torus. Then there exists an open neighbourhood U of the torus $\mathcal{F}_m$ and a diffeomorphism*

$$(\theta_1, \ldots, \theta_n, t, a_2, \ldots, a_n) : U \to \mathbb{T}^n \times B^n,$$

where t is a defining function for Z, such that

$$\omega|_U = \frac{c}{t} d\theta_1 \wedge dt + \sum_{i=2}^{n} d\theta_i \wedge dp_i.$$

Moreover, the functions $t, p_2, \ldots, p_n$ depend only on F. The number c is the modular period of the component of Z containing m.

The S^1-valued functions

$$\theta_1, \ldots, \theta_r$$

are called angle coordinates *and the $\mathbb{R}$-valued functions*

$$t, a_2, \ldots, a_r$$

are called action coordinates*.*

1.1.4 Chapter 5: Action-angle coordinates for non-commutative b-integrable systems

The action-angle coordinate theorem for non-commutative b-integrable systems is proved. We have published this result in [KMb] (joint work with Eva Miranda).

Theorem B (Action-angle coordinates for non-commutative b-integrable systems)**.** *Let (M,ω) be a b-symplectic manifold with critical hypersurface Z. Let F be a non-commutative b-integrable system on (M,ω) of rank r and let $m \in Z$ be a regular point of the system lying inside the critical hypersurface. Assume that the integral manifold $\mathcal{F}_m$ containing m is compact, i.e. a Liouville torus. Then there exists an open neighbourhood U of the torus $\mathcal{F}_m$ and a diffeomorphism*

$$(\theta_1,\ldots,\theta_r,t,p_2,\ldots,p_r,x_1,\ldots,x_\ell,y_1\ldots,y_\ell): U \to \mathbb{T}^r \times B^s,$$

where $\ell = n-r = \frac{s-r}{2}$ and t is a defining function of Z, such that

$$\omega|_U = \frac{c}{t} d\theta_1 \wedge dt + \sum_{i=2}^{r} d\theta_i \wedge dp_i + \sum_{k=1}^{\ell} dx_k \wedge dy_k.$$

Moreover, the functions $f_1,\ldots,f_s$ depend on $t,p_2,\ldots,p_r,x_1,\ldots,x_\ell,y_1\ldots,y_\ell$ only. The number c is the modular period of the component of Z containing m.

The S^1-valued functions

$$\theta_1,\ldots,\theta_r$$

are called angle coordinates*, the $\mathbb{R}$-valued functions*

$$t,p_2,\ldots,p_r$$

are called action coordinates *and the remaining $\mathbb{R}$-valued functions*

$$x_1,\ldots,x_\ell,y_1\ldots,y_\ell$$

are called transverse coordinates.

1.1.5 Chapter 6: Cotangent models and examples of integrable systems

The action-angle coordinate theorems on symplectic and b-symplectic manifolds can be formulated in the language of cotangent lifts. We define the appropriate cotangent model for b-symplectic manifolds, which we call the *twisted b-cotangent lift.* Then a b-integrable system can be viewed semilocally as the twisted b-cotangent lift of the torus acting on itself by translations. Using the terminology introduced in the chapter, we obtain the following theorem:

Theorem C. *Let $F = (f_1, \dots, f_n)$ be a b-integrable system on the b-symplectic manifold (M, ω). Then semilocally around a regular Liouville torus $\mathcal{T}$, which lies inside the exceptional hypersurface Z of M, the system is equivalent to the twisted b-cotangent lift model $(T^*\mathbb{T}^n)_{tw,c}$ restricted to a neighbourhood of $(T^*\mathbb{T}^n)_0$. Here c is the modular period of the connected component of Z containing $\mathcal{T}$.*

The results of this chapter were published in [KMa] (joint work with Eva Miranda).

1.1.6 Chapter 7: KAM Theory for b-integrable systems

We prove a KAM result for b-integrable systems, which extends the classical KAM result known for symplectic manifolds. It tells us that tori whose frequency vector satisfies a certain numerical ("Diophantine") condition *survive* under perturbations of the form specified in the theorem. This result was published in [KMS] (joint work with Eva Miranda and Geoffrey Scott).

Theorem D (KAM Theorem for b-symplectic manifolds)**.** *Let $\mathbb{T}^n \times B_r^n$ be endowed with standard coordinates (φ, y) and the b-symplectic structure* (7.2)*. Consider a b-function*

$$H = k \log |y_1| + h(y)$$

on this manifold, where h is analytic. Let y_0 be a point in B_r^n with first component equal to zero, so that the corresponding level set $\mathbb{T}^n \times \{y_0\}$ lies inside the critical hypersurface Z.

Assume that the frequency map

$$\tilde{\omega} : B_r^n \to \mathbb{R}^{n-1}, \quad \tilde{\omega}(y) := \frac{\partial h}{\partial \tilde{y}}(y)$$

has a Diophantine value $\tilde{\omega} := \tilde{\omega}(y_0)$ at $y_0 \in B^n$ and that it is non-degenerate at y_0 in the sense that the Jacobian $\frac{\partial \tilde{\omega}}{\partial \tilde{y}}(y_0)$ is regular.

Then the torus $\mathbb{T}^n \times \{y_0\}$ persists under sufficiently small perturbations of H which have the form mentioned above, i.e. they are given by ϵP, where $\epsilon \in \mathbb{R}$ and $P \in {}^b C^\infty(\mathbb{T}^n \times B_r^n)$ has the form

$$\begin{aligned} P(\varphi, y) &= k' \log |y_1| + f(\varphi, y) \\ f(\varphi, y) &= f_1(\tilde{\varphi}, y) + y_1 f_2(\varphi, y) + f_3(\varphi_1, y_1). \end{aligned}$$

More precisely, if $|\epsilon|$ is sufficiently small, then the perturbed system

$$H_\epsilon = H + \epsilon P$$

admits an invariant torus $\mathcal{T}$.

Moreover, there exists a diffeomorphism $\mathbb{T}^n \to \mathcal{T}$ close[3] *to the identity taking the flow γ^t of the perturbed system on $\mathcal{T}$ to the linear flow on $\mathbb{T}^n$ with frequency vector*

$$\left(\frac{k+\epsilon k'}{c}, \tilde{\omega}\right).$$

1.1.7 Chapter 8: Examples of singular symplectic structures in physics

We discuss several examples where non-canonical transformations are typically employed to study and/or solve problems in celestial mechanics. These are the Levi-Civita and KS transformations in the Kepler problem, the transformation due to McGehee for triple collisions in the three body problem and the McGehee coordinates in the elliptic restricted three-body problem. An interesting feature of these examples is that the critical set of the b-symplectic/folded symplectic structure is identified with what is known in celestial mechanics as the collision set or "line at infinity". These results were published in [DKM] (joint work with Amadeu Delshams and Eva Miranda).

1.2 Publications resulting from this thesis

We have published the results of this thesis in the following journals:

A. Kiesenhofer, E. Miranda, G. Scott, *Action-angle variables and a KAM theorem for b-Poisson manifolds*, J. Math. Pures Appl. (9) 105 (2016), no. 1, 66–85.

A. Delshams, A. Kiesenhofer, E. Miranda, *Examples of integrable and non-integrable systems on singular symplectic manifolds*, Journal of Geometry and Physics, 2016, DOI 10.1016/j.geomphys.2016.06.011.

A. Kiesenhofer, E. Miranda, *Cotangent models for integrable systems in symplectic and b-Poisson manifolds*, Communications in Mathematical Physics, 2016, 1-23, DOI 10.1007/s00220-016-2720-x.

[3] By saying that the diffeomorphism is "ϵ-close to the identity" we mean that, for given H, P and r, there is a constant C such that $\|\psi - \mathrm{id}\| < C\epsilon$.

A. Kiesenhofer, E. Miranda, *Non-commutative integrable systems on b-symplectic manifolds*, to appear in Journal of Regular and Chaotic Dynamics, Volume 21, Issue 6 of 2016.

Chapter 2

Preliminaries

We begin by reviewing some basic facts about Poisson manifolds and introduce the class of b-Poisson manifolds, which are the main object of study of this thesis.

If not stated otherwise, all manifolds and functions are assumed to be *smooth*.

2.1 Background on Poisson manifolds

A **Poisson structure** (or Poisson bracket) on a manifold M is given by a Lie bracket $\{\ ,\ \}$ on the space of functions on M, which moreover is a derivation in each argument (Leibniz identity):

$$\begin{gathered}\{\ ,\ \} : \mathbb{C}^\infty(M) \times \mathbb{C}^\infty(M) \to \mathbb{C}^\infty(M) \\ \{f, gh\} = g\{f, h\} + h\{f, g\}.\end{gathered}$$

Recall that a Lie bracket is, by definition, bilinear and antisymmetric, and satisfies the Jacobi identity:

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0.$$

Poisson structures as bivector fields. We can interpret the bracket $\{\ ,\ \}$ as a bivector field, i.e. a section Π of the bundle $TM \wedge TM$, by defining

$$\Pi : T^*M \times T^*M \to \mathbb{R} : (df, dg) \mapsto \{f, g\}$$

for exact one-forms df, dg and extending linearly to the whole cotangent space.

Not every bivector field defines a Poisson structure: Given $\Pi \in \Gamma(TM \wedge TM)$, the induced bracket of functions $\{\,,\,\}$ is obviously bilinear, antisymmetric and satisfies the Leibniz identity. However, it does not automatically satisfy the Jacobi identity. For the latter, he condition that has to be demanded for the bivector field Π is that the Schouten bracket $[\Pi, \Pi]$ is zero[1].

Anchor map. Associated to Π there is a bundle morphism

$$\# : T^*M \to TM : \nu \mapsto \Pi(\nu, \cdot).$$

called the anchor map. The rank of the anchor map at a point $x \in M$ is called the rank of Π at x. If the rank of Π is constant on M, we call Π a regular Poisson structure on M. Note that the rank is always an even number.

Example 2.1.1. We give some examples of Poisson structures:

- On any manifold we can define the zero Poisson structure $\Pi = 0$.
- Symplectic manifolds are regular Poisson manifolds with full rank: The Poisson bracket can be defined via

$$\{f, g\} := \omega(X_f, X_g), \qquad f, g \in C^\infty(M) \tag{2.1}$$

  where for a function f the vector field X_f is the Hamiltonian vector field of f defined by $\iota_{X_f}\omega = -df$. The closedness condition of the symplectic form corresponds to the Jacobi identity of the Poisson bracket.

- The dual $\mathfrak{g}^*$ of any Lie algebra $\mathfrak{g}$ carries a natural Poisson structure:

$$\{f, g\}(\eta) = \langle \eta, [df_\eta, dg_\eta] \rangle$$

  where f, g are functions $\mathfrak{g}^* \longrightarrow \mathbb{R}$ and $\eta \in \mathfrak{g}^*$. Here, df_η and dg_η are naturally identified with elements of $\mathfrak{g}$ and $[,]$ denotes the Lie algebra bracket on $\mathfrak{g}$.

- Fix a smooth function $K \in \mathcal{C}^\infty(\mathbb{R}^3)$ and consider the bracket

$$\{f, g\}_K = \det(df, dg, dK)$$

  for $f, g \in \mathcal{C}^\infty(\mathbb{R}^3)$.

[1]The Schouten bracket is a generalization of the Lie bracket to multivector fields, see e.g. [DZ] for a definition.

Hamiltonian vector fields. To a function $f \in \mathbb{C}^\infty(M)$ we associate a vector field
$$X_f := \#(df) = \Pi(f, \cdot) \in \Gamma(TM),$$
the **Hamiltonian vector field** of f. The set of Hamiltonian vector fields defines a smooth distribution in the Stefan-Sussmann sense, i.e. for each point $p \in M$ there is an assigned vector space $D_p \subset T_pM$. The dimension of these vector spaces might vary from point to point, in which case we talk of a *singular* distributions.

The distribution of Hamiltonian vector fields is smooth and involutive: the Lie bracket of two Hamiltonian vector fields is again a Hamiltonian vector field
$$[X_f, X_g] = X_{\{f,g\}},$$
and hence by Stefan-Sussmann's theorem (see [DZ], Theorem 1.5.5) the distribution is integrable. The corresponding, possibly singular, foliation[2] has leaves which carry a natural symplectic structure: on a leaf L define the two-form ω_L by
$$\omega_L(X_f, X_g) := \{f, g\}.$$
We refer to this (usually singular) foliation as the **symplectic foliation** associated to Π.

Assume that Π has full rank at some point in M. Then M must have even dimension, $\dim M = 2n$. We call points where Π has full rank *non-degenerate*. Note that the set of non-degenerate points is the complement of the zero set of the map $\Lambda^n\Pi : M \to \Lambda^n TM$ and is an open symplectic submanifold of M.

For a Poisson structure which does not have full rank everywhere, we can distinguish the following important class of functions:

Definition 2.1.2. A function f whose Hamiltonian vector field X_f is zero is called a **Casimir function** of the Poisson structure.

Poisson vector fields. Hamiltonian vector fields have the property that they preserve the Poisson structure, $\mathcal{L}_{X_f}\Pi = 0$, since by the standard formula for the Lie derivative $\mathcal{L}_{X_f}$ of a tensor field and the Jacobi identity for the

[2]We use the Stefan-Sussmann definition for singular foliations, i.e. there is a partition of M into immersed submanifolds called leaves such that around every point $x \in M$ there is a chart $(x_1, \ldots, x_n)$ such that the leaf through x is locally given by $\{x_1 = \ldots = x_m = 0\}$ $(m \leq n)$ and every level set $\{x_1 = c_1, \ldots, x_m = c_m\}$ is contained in some leaf. The foliation is called *regular* if all leaves have the same dimension.

Poisson bracket we have

$$(\mathcal{L}_{X_f}\Pi)(g,h) = X_f(\Pi(g,h)) - \Pi(X_f(g),h) - \Pi(g,X_f(h)) = 0.$$

Definition 2.1.3. A vector field $v \in \Gamma(TM)$ satisfying

$$\mathcal{L}_v\Pi = 0$$

is called a **Poisson vector field**.

2.1.1 Weinstein's splitting theorem

The local study of Poisson structures is based on the following theorem [We]:

Theorem 2.1.4 (Weinstein)**.** *Let (M^N, Π) be a Poisson manifold of rank $2k$ at a point $p \in M$. Then on a neighborhood of p there exist coordinates*

$$(x_1, y_1, \dots, x_k, y_k, z_1, \dots, z_{N-2k})$$

centered at p such that the Poisson structure can be written as

$$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{i,j=1}^{N-2k} f_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, \tag{2.2}$$

where f_{ij} are functions which depend only on the variables $(z_1, \dots, z_{N-2k})$ and which vanish at the origin.

This means that locally a Poisson manifold is the product of a symplectic manifold and a manifold whose Poisson structure has rank zero at the point in consideration. We also refer to the rank zero part as the *transverse part* of the Poisson structure (at p).

2.2 b-Poisson manifolds

We are interested in a class of Poisson manifolds whose Poisson structure has full rank away from a hypersurface $Z \subset M$ and its failure to be regular is "controlled" in the following way:

Definition 2.2.1. Let (M^{2n}, Π) be an oriented Poisson manifold. If the map

$$p \in M \mapsto (\Pi(p))^n \in \Lambda^{2n}(TM)$$

is transverse to the zero section, then Π is called a **b-Poisson structure** on M. The hypersurface Z where the multivectorfield Π^n vanishes,

$$Z = \{p \in M | (\Pi(p))^n = 0\}$$

is called the **critical hypersurface** of Π. The pair (M, Π) is called a **b-Poisson manifold**.

Remark. Observe that the transversality condition in the definition above is equivalent to 0 being a regular value of the map $p \in M \mapsto (\Pi(p))^n \in \Lambda^{2n}(TM)$. Moreover, note that, since M is oriented, the hypersurface Z has a *global* defining function given by dividing $(\Pi(p))^n$ by a non-vanishining section of $\Lambda^{2n}(TM)$.

On Z the rank of the Poisson structure is less than $2n$ by definition. In view of the local splitting given by Theorem 2.1.4, the transversality condition implies that the rank on Z is exactly $2n-2$.

In the following sections we collect the most important results on b-Poisson manifolds, which will provide the basis for the main results of this thesis. Before this, we give some examples of b-Poisson structures:

2.2.1 Basic examples

Example 2.2.2. Let (N^{2n+1}, π) be a regular corank-1 Poisson manifold and let X be a Poisson vector field on N that is transverse to the symplectic leaves of π. Let $f : \mathbb{R} \to \mathbb{R}$ a smooth function. The bivector field

$$\Pi = f(x)\frac{\partial}{\partial x} \wedge X + \pi$$

is a b-Poisson structure on $\mathbb{R} \times N$ if the function f vanishes linearly. If this is the case, then the critical hypersurface consists of the union of as many copies of N as zeros of f.

In this example, N^{2n+1} is the critical hypersurface of the b-Poisson manifold and Π induces on N the corank one Poisson structure π. It is a general fact that the critical hypersurface of a b-Poisson manifold naturally inherits a corank-one Poisson structure.

This example is *generic* in the sense that it provides the semilocal model for a b-Poisson structure in a neighbourhood of the critical hypersurface Z.

Example 2.2.3. (Radko sphere) A simple example of a compact b-Poisson manifold is S^2 endowed with

$$\Pi = h\frac{\partial}{\partial h} \wedge \frac{\partial}{\partial \theta},$$

where (h, θ) are the standard height and angle coordinates. The critical hypersurface is the equator and on an neighborhood of it (i.e. semilocally) the manifold is isomorphic to the previous example setting $N = S^1$, $\pi = 0$, $X = \frac{\partial}{\partial \theta}$ and $f(h) = h$.

2.2.2 Historical remarks

The initial motivation to consider b-symplectic manifolds comes from the study of differential calculus on manifolds with boundary initiated by Melrose [Me] and its connection to deformation quantization, which was studied by Nest [NT]. Melrose's b-calculus is also the origin of the term "b-symplectic" — the letter b comes from the word "boundary". In our definition of b-Poisson manifolds the boundary is replaced by a distinguished hypersurface $Z \subset M$.

In the two-dimensional case, b-Poisson structures in were studied and classified by Radko [R]. The systematic study of b-Poisson manifolds in any dimension started with the work of Guillemin, Miranda and Pires [GMP11, GMP14] and has attracted the attention of other mathematicians who contributed important results on the geometry and topology of these manifolds, see for instance [GLPR, FMM, MO13, GMPS13, P]. Our recent publications [KMS, KMa, KMb, DKM] extend this development and contribute results about the *dynamics* of b-symplectic manifolds as well as examples coming from physics.

2.2.3 The b-tangent and b-cotangent bundles

Let (M^{2n}, Z, Π) be a b-Poisson manifold. On $M\backslash Z$ the Poisson structure Π induces a symplectic form, which "goes to infinity" when we approach Z. The formal approach of defining differential forms with this type of singularity is the concept of b-de Rham forms.

We start with with following definitions:

Definition 2.2.4. A b-**manifold** is a pair (M^N, Z) of an oriented manifold

M and an oriented hypersurface $Z \subset M$. A b**-map** is a map

$$f : (M_1, Z_1) \to (M_2, Z_2)$$

transverse to Z_2 and such that $f^{-1}(Z_2) = Z_1$. The b**-category** is the category whose objects are b-manifolds and morphisms are b-maps.

Definition 2.2.5. A b**-vector field** on a b-manifold (M, Z) is a vector field which is tangent to Z at every point $p \in Z$.

These vector fields form a Lie subalgebra of the algebra of all vector fields on M. Moreover, if f is a local defining function for Z on some open set $U \subset M$ with non-empty intersection $U \cap Z$, and $(f, x_2, \ldots, x_N)$ is a chart on U, then the set of b-vector fields on U is a free $C^\infty(U)$-module with basis

$$(f\frac{\partial}{\partial f}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_N}). \tag{2.3}$$

Hence the sheaf of b-vector fields on M is a locally free C^∞-module and therefore it is given by the sections of a vector bundle on M. We call this vector bundle the b**-tangent bundle** and denote it ${}^bTM$.

b**-tangent spaces.** We want to understand the fibers of the vector bundle ${}^bTM$, i.e. the b-tangent spaces ${}^bT_pM$ where $p \in M$. At points $p \in M \backslash Z$, the b-tangent space coincides with the usual tangent space ${}^bT_pM = T_pM$. On the other hand, restricting a b-vector field to Z yields a vector field on Z in a natural way. The corresponding vector bundle morphism

$$\psi_Z :^b TM|_Z \to TZ. \tag{2.4}$$

is surjective, as we can see e.g. by considering the local frame given in Equation (2.3). Its kernel is a line bundle and a nonvanishing section of it is called a **normal** b**-vector field** of the b-manifold (M, Z). In the local frame (2.3), the kernel of ψ_Z is spanned by the section $(f\frac{\partial}{\partial f})|_Z$.

b**-cotangent bundle.** We define the b-cotangent bundle ${}^bT^*M$ of M to be the vector bundle dual to ${}^bTM$. The discussion in the previous paragraph implies that for $p \in M \backslash Z$, the b-cotangent space coincides with the usual cotangent space: ${}^bT^*_pM = T^*_pM$. At points $p \in Z$, the dual of the map (2.4) yields an injective morphism

$$\psi_Z^* : T_p^*Z \to {}^bT_p^*M \tag{2.5}$$

whose image is $\left\{\nu_p \in {}^b T_p^* M | \nu_p(w_p) = 0,\, p \in Z\right\}$, where w is a normal b-vector field of (M, Z) as defined above.

Let f be a global defining function of Z (which exists due to the remark after Definition 2.2.1). Then $\mu := \frac{df}{f} \in \Omega^1(M \backslash Z)$ defines a (smooth) one-form on $M \backslash Z$. We denote by $\langle \cdot, \cdot \rangle$ the canonical pairing of forms and vector fields. For any b-vector field v on M the pairing $\langle v, \mu \rangle \in C^\infty(M \backslash Z)$ extends smoothly over Z and hence μ itself extends smoothly over Z as a section of ${}^b T^* M$. It therefore defines a b-one-form on M, which we write as $\frac{df}{f}$, understanding the notation in the way just described. This b-one-form has the property that its pairing with the normal b-vector field $w = f \frac{\partial}{\partial f}$ is nonzero, $\mu_p(w_p) = 1$ for $p \in Z$, and hence the discussion of the previous paragraph entails the following splitting:

$$ {}^b T_p^* M = T_p^* Z + \operatorname{span}\left\{\frac{df}{f}\right\}. $$

2.2.4 The b-de Rham complex

Having introduced b-forms, we can define forms of higher order by the usual recipe:

Definition 2.2.6. For $k > 0$ we define the space of **b-de Rham k-forms** as the sections of the vector bundle $\Lambda^k({}^b T^* M)$ and denote it ${}^b\Omega^k(M)$.

The classical space of de Rham k-forms sits inside the space of b-de Rham k-forms. More precisely, there is an injective sheaf morphism

$$ \phi : \Omega^k \longrightarrow {}^b\Omega^k $$

defined fiberwise in the following way: given a k-form $\mu \in \Omega^k(M)$, we set

$$ \phi(\mu)_p := \begin{cases} \mu_p & \text{at } p \in M \backslash Z \\ (i^*\mu)_p \in \Lambda^k(T_p^* Z) \subset \Lambda^k({}^b T_p^* M) & \text{at } p \in Z \end{cases} $$

where $i : Z \hookrightarrow M$ is the inclusion map and $\Lambda^k(T_p^* Z)$ is viewed as a a subset of $\Lambda^k({}^b T_p^* M)$ by the injection ψ_Z^* given in Equation (2.5).

Remark. Note that, in particular, if f is a defining function of Z, the recipe above takes the de Rham form df to a b-de Rham form which is zero on Z, since $(i^* df)_p = 0$ for $p \in Z$. Hence evaluated at a point in Z the map which takes sections of T^*M to sections of ${}^bT^*M$ is not injective. However, as a sheaf morphism it is injective, which is why we can say that $\Omega(M)$ is contained in ${}^b\Omega(M)$. The injectivity as a map of sheaves is used in the b-Mazzeo-Melrose theorem to construct a short exact sequence (see Section 2.2.7 and in particular Theorem 2.2.17), yielding a long exact sequence which allows the computation of the so-called b-cohomology.

With these conventions it is easy to see that, having fixed a defining function f, every b-de Rham k-form ($k > 0$) can be written as

$$\omega = \alpha \wedge \frac{df}{f} + \beta, \text{ with } \alpha \in \Omega^{k-1}(M) \text{ and } \beta \in \Omega^k(M). \tag{2.6}$$

While α and β are not unique, it is easy to show that evaluated at a point $p \in Z$, α_p and β_p are unique (Proposition 5 in [GMP11]).

The decomposition (2.6) enables us to extend the exterior d operator to ${}^b\Omega(M)$ by setting

$$d\omega = d\alpha \wedge \frac{df}{f} + d\beta.$$

The right hand side is well defined and agrees with the usual exterior d operator on $M \setminus Z$ and also extends smoothly over M as a section of $\Lambda^{k+1}({}^bT^*M)$. Note that $d^2 = 0$, which allows us to define the following complex of b-forms, which is called the ***b*-de Rham complex**:

$$0 \to \mathbb{R} \to {}^b\Omega^0(M) \xrightarrow{d} {}^b\Omega^1(M) \xrightarrow{d} {}^b\Omega^2(M) \xrightarrow{d} \ldots \to 0 \tag{2.7}$$

Here ${}^b\Omega^0(M) = {}^bC^\infty(M)$ is the set of b-functions which we will define below.

Moreover, we can define the Lie derivative of b-forms via the **Cartan formula**:

Definition 2.2.7 (Lie derivative of b-forms). Let $\omega \in {}^b\Omega^k(M)$. Then we define the Lie derivative with respect to a b-vector field X as

$$\mathcal{L}_X\omega = \iota_X(d\omega) + d(\iota_X\omega) \in {}^b\Omega^k(M). \tag{2.8}$$

Note that if $\omega = \alpha \wedge \frac{df}{f} + \beta \in \Omega^k(M)$, then another way to write the Lie derivative is

$$\mathcal{L}_X\omega := \mathcal{L}_X\alpha \wedge \frac{df}{f} + \alpha \wedge d(X(\log|f|)) + \mathcal{L}_X\beta,$$

so in particular the Lie derivative for b-forms defined above is a generalization of , the usual Lie derivative for smooth forms (i.e. $\alpha = 0$).

b-functions. In order for the b-de Rham complex to admit a Poincaré lemma, it is convenient to enlarge the set of smooth functions and consider the set of *b-functions* ${}^bC^\infty(M)$, which consists of functions with values in $\mathbb{R} \cup \{\infty\}$ of the form

$$c\log|f| + g,$$

where $c \in \mathbb{R}$, f is a defining function for Z, and g is a smooth function. We define the differential operator d on this space in the obvious way:

$$d(c\log|f| + g) := \frac{cdf}{f} + dg \in {}^b\Omega^1(M),$$

where dg is the standard de Rham derivative.

2.2.5 b-symplectic manifolds

Definition 2.2.8. Let (M^{2n}, Z) be a b-manifold and $\omega \in {}^b\Omega^2(M)$ a closed b-form. We say that ω is **b-symplectic** if ω_p is of maximal rank as an element of $\Lambda^2({}^bT_p^*M)$ for all $p \in M$.

Here "maximal rank" at a point p means that the linear map

$$ {}^bT_pM \to {}^bT_p^*M : u \mapsto \omega_p(u, \cdot) \tag{2.9}$$

has maximal rank $2n$, i.e. it is an isomorphism.

Dual b-Poisson structure. We have already mentioned that given a Poisson structure Π, we can dualize it on the set of points where it has maximal rank to obtain a symplectic structure there.

Given a b-symplectic structure ω we can apply the analogous procedure to the isomorphism between the b-tangent and b-cotangent bundle given in Equation (2.9). By inverting this isomorphism we obtain a map

$${}^bT_p^*M \to^b T_pM$$

and since ${}^bTM$ sits inside TM this yields a map

$${}^bT_p^*M \to^b T_pM \hookrightarrow T_pM.$$

Using this map we can associate to a b-symplectic form ω a bivector field $\Pi \in \Gamma(\Lambda^2 TM)$, which we call the dual of ω.

In [GMP11] (Proposition 20) the authors prove the following:

Proposition 2.2.9. *A two-form ω on a b-manifold (M, Z) is b-symplectic if and only if its dual bivector field Π is a b-Poisson structure.*

In short, "b-Poisson equals b-symplectic".

b-Hamiltonian vector fields. The non-degeneracy of the b-symplectic form allows us to define the b-Hamiltonian vector field X_f of a b-function $f \in^b C^\infty(M)$ intrinsically:

$$\iota_{X_f}\omega = -df.$$

This is in complete analogy to the symplectic case. Note that the same vector field is obtained if we use the dual b-Poisson structure Π and define

$$X_f = \Pi(df, \cdot).$$

Obviously, the flow of a b-Hamiltonian vector field preserves the b-symplectic form and hence the Poisson structure, so b-Hamiltonian vector fields are in particular Poisson vector fields.

b-Darboux theorem. We have already stated the Weinstein splitting theorem for general Poisson manifolds (Theorem 2.1.4). In the case of b-Poisson manifolds, if we consider the splitting around a point $p \in Z$ the transverse part is 2-dimensional. For the dual b-symplectic structure ω this means that

$$\omega = \omega_L + (\Pi^T)^\sharp$$

where ω_L is the symplectic form on the symplectic leaf through the point $p \in Z$ and $(\Pi^T)^\sharp$ is the dual to a b-Poisson structure on a 2-dimensional manifold which has rank zero on Z.

More precisely, one can prove the following local normal form result that is the analogue to the Darboux theorem for symplectic manifolds ([GMP11]) and which is therefore called the "b-Darboux theorem":

Theorem 2.2.10 (**b-Darboux theorem**)**.** *Let (M, Z, ω) be a b-symplectic manifold. Then, on a neighborhood of a point $p \in Z$, there exist coordinates $(z, t, x_1, y_1, \ldots, x_{n-1}, y_{n-1})$ centered at p such that*

$$\omega = \frac{1}{t}\, dz \wedge dt + \sum_{i=1}^{n-1} dx_i \wedge dy_i. \tag{2.10}$$

Remark. As is clear from the proof of the b-Darboux theorem in [GMP14], we can specify a particular local defining function t of the critical hypersurface around m and complete it to a coordinate system $(z, t, x_1, y_1, \ldots, x_{n-1}, y_{n-1})$ such that the above holds.

Note that on the chart given in the theorem, the symplectic foliation of Π has the following form: it contains two open leaves where the Poisson structure has full rank — the upper and lower half spaces given by $t > 0$ and $t < 0$ — and the union of the remaining leaves is the hyperplane $Z = \{t = 0\}$, where Π^n vanishes. The leaves inside Z are $(2n-2)$-dimensional subspaces given by the level sets of z. Restricting the b-Poisson structure Π to the critical hypersurface Z gives a regular corank one Poisson structure $\tilde{\Pi}$, which in local b-Darboux coordinates has the form $\sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$.

In most parts of this thesis we will work with the b-symplectic viewpoint, i.e. in the language of b-forms instead of Poisson structures.

2.2.6 The topology of the critical hypersurface

The modular vector field

For any Poisson manifold (M, Π) with volume form Ω we can ask inhowfar Hamiltonian vector fields preserve Ω. The *modular vector field* gives information about this:

Definition 2.2.11. The map

$$f \mapsto \frac{\mathcal{L}_{X_f}\Omega}{\Omega} \tag{2.11}$$

is a derivation on M, hence a vector field, called the **modular vector field** of (M, Π, Ω). If Π and Ω are understood from the context we will simply denote the modular vector field by v_{mod}.

It can be shown that v_{mod} is a Poisson vector field, i.e. $\mathcal{L}_{v_{\mathrm{mod}}}\Pi = 0$. Moreover, if we consider the class of v_{mod} in the quotient of Poisson vector fields modulo Hamiltonian vector fields (i.e. as a class in the Poisson cohomology of M), then it is independent of the volume form chosen. This class is called the **modular class of** (M, Π). A Poisson manifold is called **unimodular** if its modular class is zero.

We now consider the case of a b-Poisson manifold with critical hypersurface Z. We compute the modular vector field locally on a b-Darboux chart

$$(z, t, x_1, y_1, \dots, x_{n-1}, y_{n-1})$$

around a point in Z. On this chart the b-symplectic form takes the form given in Equation (2.10) and we choose the volume form with local expression

$$\Omega = dz \wedge dt \wedge dx_1 \wedge dy_1 \wedge \ldots \wedge dx_{n-1} \wedge dy_{n-1}.$$

Then we can explicitly compute the map in Equation (2.11): For the coordinate function $f = x_1$ we have $X_f = \frac{\partial}{\partial y_1}$ and therefore the Lie derivative $\mathcal{L}_{X_f}\Omega$ is zero; in the same way we see that all other coordinate functions except for t get mapped to zero. The latter has Hamiltonian vector field $X_t = z\frac{\partial}{\partial z}$ and therefore $\mathcal{L}_{X_f}\Omega = \Omega$. Hence the modular vector field on the chart is given by $\frac{\partial}{\partial t}$. Note that this vector field is b-Hamiltonian, $\frac{\partial}{\partial t} = X_{-\log|z|}$, and therefore in particular Poisson. Moreover, it is transverse to the symplectic leaves inside Z. The last two properties are not affected if we add a Hamiltonian vector field and therefore they do not depend on our choice of Ω on the chart. Hence covering Z by such charts, we obtain the following global result:

Proposition 2.2.12. *The modular vector field on a b-Poisson manifold is a Poisson vector field and transverse to the symplectic foliation on the critical hypersurface.*

Remark. The above proposition implies that the critical hypersurface is **cosymplectic**, see [FMM, GMP11] for the definition and properties. An important contribution to the study of cosymplectic manifolds is [CFL].

The symplectic foliation on Z

Let (M, Z, ω) be a b-symplectic manifold. The corank one foliation $\mathcal{F}$ induced on Z has some special topological properties: It is transversely orientable; as mentioned above a transverse vector field is given by the modular vector field. This implies that $\mathcal{F}$ has a global defining one-form[3], i.e. a one-form $\alpha \in \Omega^1(Z)$ such that $\ker \alpha_p = T_pL_p$, where L_p is the leaf of $\mathcal{F}$ through p.

A more elaborate discussion shows that the defining one-form can be chosen to be closed [GMP11] (see Proposition 18 therein). Therefore, in the case where $\mathcal{F}$ has a compact leaf, a version of *Reeb's stability theorem* implies that $\mathcal{F}$ is a fibration over a circle:

Proposition 2.2.13. *Let (M, Z, ω) be a b-symplectic manifold. Assume that Z is compact and connected and that the induced corank one foliation on Z has a compact leaf L. Then Z is the mapping torus*

$$Z \cong (L \times [0,1])/_{(x,0)\sim(\phi(x),1)},$$

of the diffeomorphism $\phi : L \to L$ given by the holonomy map of the fibration

[3] We can construct α by taking a family of local defining one-forms $(\alpha_U)_{U\in\mathcal{U}}$, where $\mathcal{U}$ is an open cover of Z, and demanding that α_U satisfies $\alpha_U(v|_U) = 1$ on U. To obtain α we glue these local one-forms together by using a partition of unity.

over S^1. In particular, all the symplectic leaves inside Z are symplectomorphic.

In the transverse direction to the symplectic leaves, all the modular vector fields flow with the same speed. This allows the following definition:

Definition 2.2.14 (Modular period). Taking any modular vector field u^{Ω}_{mod}, the **modular period** of a connected component Z' of Z with a compact leaf is the number k such that Z' is the mapping torus

$$Z' = (\mathcal{L} \times [0,k])/_{(x,0)\sim(\phi(x),k)},$$

and the time-t flow of u^{Ω}_{mod} is translation by t in the $[0,k]$ factor above.

2.2.7 b-Cohomology

Given a b-manifold (M,Z), we can consider the usual cohomology theories for the underlying manifold M, such as de Rham cohomology and Poisson cohomology, which correspond respectively to de Rham forms and to multivector fields. On the other hand, we can define cohomology theories based on the notions of b-forms and b-multivector fields.

Definition 2.2.15. The **b-de Rham cohomology**, or b-cohomology for short, is the cohomology of the b-de Rham complex given by Equation (2.7).

Poisson cohomology. On the other hand, for a general Poisson manifold (M,Π), the Poisson structure induces a differential operator

$$d_\pi = [\Pi, \cdot\,]$$

on the graded algebra of multivector fields $(\Lambda^k TM)_{k\in\mathbb{N}}$, where $[\,,\,]$ is the Schouten bracket. The cohomology of this complex is called the **Poisson cohomology** and denoted $H^*_\Pi(M)$.

Restricting the operator d_π to b-multivector fields, i.e. sections of the bundle $\Lambda^k({}^bTM), k \in \mathbb{N}$, we obtain another differential complex whose cohomology is called the **b-Poisson cohomology** and denoted ${}^bH^*_\Pi(M)$.

For symplectic manifolds, the Poisson cohomology is isomorphic to the de Rham cohomology. Similarly, for b-symplectic manifolds we have the following result proved in [GMP14]:

Proposition 2.2.16. *The b-Poisson cohomology is isomorphic to the b-de Rham cohomology.*

Mazzeo-Melrose theorem. The Mazzeo-Melrose theorem allows us to compute the b-de Rham cohomology of (M, Z) based on the ordinary de Rham cohomology of M and Z. The theorem we present here was proved in [GMP11]. Its original version in the setting of manifolds with boundary goes back to [Me].

Theorem 2.2.17 (b**-Mazzeo-Melrose theorem**). *Let (M, Z) be a b-manifold with $Z \overset{i}{\hookrightarrow} M$ compact. Then the b-cohomology groups of M are computable by*

$$^bH^*(M) \cong H^*(M) \oplus H^{*-1}(Z).$$

The identification is given by

$$[\omega] \mapsto [\alpha] + [\beta]$$

where we express the closed k-form ω as $\omega = \alpha \wedge \frac{df}{f} + \beta$ with closed $\alpha \in \Omega^{k-1}(M)$ and closed $\beta \in \Omega^k(M)$. In more detail, the proof uses the short exact sequence

$$0 \to \Omega^k(M) \to^b \Omega^k(M) \to \Omega^{k-1}(Z) \to 0$$

where the first arrow is simply inclusion and the second arrow maps $\omega = \alpha \wedge \frac{df}{f} + \beta$ to $i^*\alpha$. The corresponding long exact sequence

$$\cdots \to H^k(M) \to^b H^k(M) \to H^{k-1}(Z) \to H^{k-1}(M) \cdots$$

splits by an argument given in [GMP11], yielding the splitting of the b-Mazzeo Melrose theorem above.

2.3 b^m-symplectic structures and folded symplectic structures

In this section we define b^m-symplectic structures and (m-)folded symplectic structures. These singular structures and the b-symplectic structures that we have already studied in previous sections share a common feature: they are symplectic away from a singular set, which is a hypersurface, and their singularities on this surface are "controlled" in a certain way. Moreover, these structures have a simple Darboux canonical form around points on the critical set, which we will state below.

2.3.1 b^m-symplectic structures

The idea is to consider structures similar to the b-symplectic case but where the singularity is of "higher order". The formal approach uses the language of jets [Sc]:

Definition 2.3.1. Let $Z \subset M$ be a hypersurface and ι the inclusion of Z. Let C^∞ denote the sheaf of C^∞-functions on M and let $\mathcal{I}$ be the ideal sheaf of Z. Then the sheaf of m-jets at Z is defined as $\mathcal{J}^m := \iota^*(C^\infty/\mathcal{I}^{m+1})$

For j an m-jet we use the notation $f \in j$ to say that f represents j.

Definition 2.3.2 (b^m-manifold). A b^m**-manifold** is a triple (M, Z, j) where M is an oriented manifold, $Z \subset M$ an oriented hypersurface and j is an element of $\mathcal{J}^{k-1}$ that can be represented by a positively oriented[4] local defining function of Z.

We can now mathematically express what it means for a vector field to be "tangent of higher order" and thus generalize the definition of b-vector fields to higher order:

Definition 2.3.3 (b^m-vector field). A b^m**-vector field** on the b^m**-manifold** (M, Z, j) is a vector field v tangent to Z (i.e. $v_p \in T_pZ$ for $p \in Z$) such that for any $f \in j$, $v(f) \in \mathcal{I}^m$.

As in the case of b-vector fields, the sheaf of b^m-vector fields is *locally free*. As is explained in detail in [Sc], for (M, Z, j) a b^m-manifold, $p \in Z$ and $(x_1, \dots, x_n)$ a chart on a neighbourhood U of p in M with $x_1 \in j$, the vector fields

$$\left\{ x_1^k \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right\}$$

form a basis of the $C^\infty(U)$-module of b^m-vector fields on U.

The corresponding vector bundle is denoted ${}^{b^m}TM$ and we call it the b^m-tangent bundle. The dual is the b^m-cotangent bundle ${}^{b^m}T^*M$ and taking the wedge product we obtain forms of higher order, just as in the standard b-case that we have already discussed (Section 2.2.4).

[4]We say that a local defining function f is positively oriented if for a positively oriented volume form Ω of Z, $df \wedge \Omega$ is positively oriented for M.

Definition 2.3.4. A **symplectic b^m-manifold** is a pair (M^{2n}, Z) with a closed b^m-two form ω which has maximal rank at every $p \in M$.

In [GMW] a b^m-Darboux theorem is proved for these structures:

Theorem 2.3.5 (b^m**-Darboux theorem, [GMW]).** *Let ω be a b^m-symplectic form on (M^{2n}, Z) and $p \in Z$. Then we can find a coordinate chart $(x_1, y_1, \ldots, x_n, y_n)$ centered at p such that the hypersurface Z is locally defined by $y_1 = 0$ and*

$$\omega = dx_1 \wedge \frac{dy_1}{y_1^m} + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

Expressed in terms of bivector fields, the Poisson structure has the following b^m-Darboux form:

$$\Pi = y_1^m \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$$

Remark. It is possible to make even further generalizations. The b^m-Poisson structures that we just introduced correspond to a singularity of A_m-type in Arnold's list of simple singularities [A, AGV]. In the same spirit we may consider other singularities in this list.

2.3.2 Folded symplectic structures

A second class of important geometrical structures that naturally arise as a model for the phase space in some problems of celestial mechanics are *folded symplectic structures*. These are closed 2-forms on even-dimensional manifolds which are non-degenerate on a dense set and satisfy a transversality condition around the hypersurface where they fail to be non-degenerate.

Definition 2.3.6. Let (M^{2n}, ω) be a manifold with ω a closed 2-form such that the map

$$p \in M \mapsto (\omega(p))^n \in \Lambda^{2n}(T^*M)$$

is transverse to the zero section, then $Z = \{p \in M | (\omega(p))^n = 0\}$ is a hypersurface and we say that ω defines a **folded symplectic structure** on (M, Z) and (M, Z, ω) is a **folded symplectic manifold**. The hypersurface Z is called **folding hypersurface**.

The normal forms of folded symplectic structures were studied by Martinet [Mart].

Theorem 2.3.7 (Folded-Darboux theorem, [Mart]). *Let ω be a folded symplectic form on (M^{2n}, Z) and $p \in Z$. Then we can find a local coordinate chart $(x_1, y_1, \ldots, x_n, y_n)$ centered at p such that the hypersurface Z is locally defined by $y_1 = 0$ and*

$$\omega = y_1 dx_1 \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

In analogy to the case of b^m-symplectic structures we can define folded structures of higher order, so called **m-folded symplectic structures** for which ω^n has singularities of A_m-type in Arnold's list of simple singularities [A]. In this case, the top wedge ω^n has a local normal form of type

$$\omega^n = y_1^m dx_1 \wedge \cdots \wedge dy_n.$$

2.4 Hamiltonian $\mathbb{T}^r$ actions on b-symplectic manifolds

Integrable systems are in close relation with Hamiltonian torus actions. On a neighbourhood of a Liouville torus, a non-commutative b-integrable system induces an effective Hamiltonian $\mathbb{T}^r$ action (see Section 5.5.1 for details). In the commutative case, the torus has dimension r equal to half the dimension of the manifold.

Hamiltonian torus actions on b-symplectic manifolds were studied in [GMPS14]. We recall the definitions and results:

Definition 2.4.1. An action of $\mathbb{T}^r$ on a b-symplectic manifold (M^{2n}, ω) is **Hamiltonian** if for all $X, Y \in \mathfrak{t}$:

- the one-form $\iota_{X^\#}\omega$ is exact, i.e., has a primitive $H_X \in {}^bC^\infty(M)$;
- $\omega(X^\#, Y^\#) = 0$.

Here, $\mathfrak{t}$ denotes the Lie algebra of $\mathbb{T}^r$ and $X^\#$ is the fundamental vector field of X. The Hamiltonian H_X of a fundamental vector field $X^\#$ is defined via the moment map $\mu : M \to \mathfrak{t}^*$ by $H_X(p) := \langle \mu(p), x \rangle$, in other words we have

$$\iota_{X^\#}\omega|_p = d\langle \mu(p), X \rangle.$$

Definition 2.4.2. A **toric action** on a b-symplectic manifold (M^{2n}, ω) is an effective Hamiltonian $\mathbb{T}^n$ action, i.e. in addition to the conditions in Definition 2.4.1 we demand that the torus has half the dimension of M and that the action is effective.

2.4.1 Modular weights

When a b-function $f \in C^\infty(M)$ is expressed as $c \log |y| + g$ locally near some point of a component Z' of Z, the number $c_{Z'}(f) := c \in \mathbb{R}$ is uniquely determined by f, even though the functions y and g are not.

Definition 2.4.3 (Modular weight). Given a Hamiltonian $\mathbb{T}^r$-action on a b-symplectic manifold, the **modular weight** of a connected component Z' of Z is the map

$$v_{Z'} : \mathfrak{t} \to \mathbb{R}$$

given by $v_{Z'}(X) = c_{Z'}(H_X)$. This map is linear and therefore we can regard it as an element of the dual of the Lie algebra $v_{Z'} \in \mathfrak{t}^*$. We denote the kernel of $v_{Z'}$ by $\mathfrak{t}_{Z'} \subset \mathfrak{t}$.

In particular, for toric actions the modular weight is always non-zero [GMPS13].

2.4.2 Splitting of Hamiltonian torus actions

The following proposition shows that, if a component Z' has non-zero modular weight with respect to a given $\mathbb{T}^r$-action, then the action splits into a $\mathbb{T}^{r-1}$-action induced by a smooth moment map on a symplectic leaf of Z' and an S^1-action transverse to the leaves induced by a log function:

Proposition 2.4.4. *Let (M, ω) be a b-symplectic manifold endowed with an effective Hamiltonian $\mathbb{T}^r$-action such that the modular weight v_{Z_i} for a connected component Z_i of the exceptional hypersurface is nonzero. Then:*

(a) In a neighborhood of Z_i there is a splitting

$$\mathfrak{t} \simeq \mathfrak{t}_{Z_i} \times \langle X \rangle,$$

which induces a splitting $\mathbb{T}^r \simeq \mathbb{T}^{r-1}_{Z_i} \times \mathbb{S}^1$. The $\mathbb{T}^{r-1}_{Z_i}$-action on Z_i induces a Hamiltonian $\mathbb{T}^{r-1}_{Z_i}$-action on $\mathcal{L}_i$. Let

$$\mu_{\mathcal{L}_i} : \mathcal{L}_i \to \mathfrak{t}^*_{Z_i}$$

be its moment map.

(b) There is a neighborhood $\mathcal{L}_i \times \mathbb{S}^1 \times (-\varepsilon, \varepsilon) \simeq \mathcal{U} \subset M$ of Z_i such that the $(\mathbb{T}^{r-1}_{Z_i} \times \mathbb{S}^1)$-action on $\mathcal{U}\backslash Z_i$ is given by

$$(g, \theta) \cdot (\ell, \rho, t) = (g \cdot \ell, \rho + \theta, t)$$

and has moment map

$$\begin{aligned} \mu_{\mathcal{U}\backslash Z_i} : \mathcal{L}_i \times \mathbb{S}^1 \times ((-\varepsilon, \varepsilon)\backslash\{0\}) &\to \mathfrak{t}^* \simeq \mathfrak{t}^*_{Z_i} \times \mathbb{R} \\ (\ell, \rho, t) &\mapsto (\mu_{\mathcal{L}_i}(\ell), c \log |t|). \end{aligned}$$

2.5 The cotangent lift for symplectic manifolds

We introduce the classical theory of cotangent lifts, which is a way of constructing Hamiltonian actions on T^*M starting with *any* group action on the base manifold M. We will generalize this concept in Chapter 6 to the b-case and use it to describe integrable systems on b-symplectic manifolds.

Let G be a Lie group and let M be any smooth manifold. Given a group action $\rho : G \times M \longrightarrow M$, we define its cotangent lift as the action on T^*M given by $\hat{\rho}_g := \rho^*_{g^{-1}}$ where $g \in G$. We then have a commuting diagram

$$\begin{tikzcd} T^*M \arrow[r, "\hat{\rho}_g"] \arrow[d, "\pi"'] & T^*M \arrow[d, "\pi"] \\ M \arrow[r, "\rho_g"] & M \end{tikzcd}$$

where π is the canonical projection from T^*M to M.

We view the cotangent bundle T^*M as a symplectic manifold endowed with the canonical symplectic form $\omega = -d\lambda$, where λ is the Liouville one-form. The latter can be defined intrinsically:

$$\langle \lambda_m, v \rangle := \langle m, (\pi_m)_*(v) \rangle \tag{2.12}$$

with $v \in T(T^*M), m \in T^*M$.

A straightforward argument given in [GS90] shows that the cotangent lift $\hat{\rho}$ is Hamiltonian with moment map $\mu : T^*M \to \mathfrak{g}^*$ given by

$$\langle \mu(m), X \rangle := \langle \lambda_m, X^\# |_m \rangle = \langle m, X^\# |_{\pi(m)} \rangle,$$

where $m \in T^*M$, X is an element of the Lie algebra $\mathfrak{g}$ and we use the same symbol $X^\#$ to denote the fundamental vector field of X generated by the action on T^*M or M.

A direct computation shows that the Liouville one-form is invariant under the action, i.e.

$$\hat{\rho}_g^* \lambda = \lambda.$$

It is well-known that invariance of λ implies equivariance of the moment map μ, meaning that

$$\mu \circ \hat{\rho}_g = Ad^*_{g^{-1}} \circ \mu.$$

A consequence is that the moment map is Poisson, see Proposition 7.1 in [DW].

We will refer to this Hamiltonian action of G on T^*M as the **symplectic cotangent lift** of the action on M.

2.6 KAM theory for symplectic manifolds

Chapter 7 of this thesis is devoted to presenting a KAM result for b-integrable systems. Here we recall the background on KAM theory for symplectic manifolds, see e.g. [De, Ch] for interesting surveys and Chapter 15 in [K].

The classical KAM theorem – named after Kolmogorov, Arnold and Moser – is a statement about the stability of integrable Hamiltonian systems on symplectic manifolds: Roughly speaking, it implies that "most" Liouville tori of an integrable system persist under sufficiently small perturbations of the Hamiltonian function of the system.

In physics, the Hamiltonian function H (or "Hamiltonian" for short) is a function on a symplectic manifold M which determines the evolution of any other function g on M via the equation

$$\dot{g} = \{g, H\}.$$

The pair of a symplectic manifold and a Hamiltonian function is called a **Hamiltonian (dynamical) system**.

An integrable system $F = (f_1, \ldots, f_n)$ on a symplectic manifold M together with a Hamiltonian function is called an **integrable Hamiltonian system** if

$$\{f_i, H\} = 0$$

for all integrals f_i. Speaking in the language of physics, the last condition corresponds to the fact that the integrals represent constants of motion.

2.6.1 Equations of motion

From the action-angle coordinate theorem we know that the manifold is semilocally around a Liouville torus symplectomorphic to the product $\mathbb{T}^n \times B^n$ with the standard symplectic structure, with coordinates (φ, y), and with the integrals $\{y_1, \ldots, y_n\}$ being the components of the projection to the B^n component. Restricting to such a neighborhood, we can write the equations of motion explicitly:

$$\begin{aligned}\dot{\varphi} &= \frac{\partial}{\partial y} H(\varphi, y),\\ \dot{y} &= -\frac{\partial}{\partial \varphi} H(\varphi, y).\end{aligned}$$

Here, the expressions in both lines are maps from M to $\mathbb{R}^n$.

In particular, if we consider the requirement $\{y_i, H\} = 0$, this means that H is independent of the angle coordinates φ. Therefore, we can write

$$H(\varphi, y) = h(y)$$

for some function $h : B^n \to \mathbb{R}$. This system evolves in a very simple way:

$$\begin{aligned}&\dot{\varphi} = \frac{\partial}{\partial y} H(\varphi, y) =: \omega(y)\\ &\dot{y} = 0.\end{aligned}$$

and has solutions of the form

$$y(t) = y_0, \qquad \varphi(t) = \varphi_0 + \omega(y_0)t,$$

i.e. the angle coordinates wrap around the torus $\mathbb{T}^n \times \{y_0\}$ in a linear fashion with frequency vector $\omega(y_0)$.

2.6.2 Diophantine condition

We write ω for the n-tuple $\omega(y_0)$. The components of ω are *rationally dependent* if

$$\omega \cdot k := \sum_{i=1}^{n} \omega_i k_i = 0 \qquad \text{for some } k \in \mathbb{Z}^n \backslash \{0\}.$$

If all linear combinations of the form above with coefficients in $\mathbb{Z}^n \backslash \{0\}$ are non-zero then the components of ω are rationally independent. In this case the motion is called **quasi-periodic** and the resulting trajectory fills the torus densely. In KAM theory a particular kind of rationally independent frequency vectors are of interest — so-called *strongly non-resonant* or *Diophantine* frequency vectors:

Definition 2.6.1 (Diophantine condition). An n-tuple $\omega \in \mathbb{R}^n$ is called *Diophantine* if there exist $L, \gamma > 0$ such that

$$|\omega \cdot k| \geq \frac{L}{|k|^\gamma} \qquad \text{for all } k \in \mathbb{Z}^n \backslash \{0\},$$

where $|k| := \sum_{i=1}^n |k_i|$.

2.6.3 KAM Theorem

Not only does rational dependence versus independence of the components of the frequency vectors result in very different kinds of motion on the tori, it also has deep implications on the behaviour of the corresponding torus under *perturbations* of the Hamiltonian system. Indeed, it can be shown that the tori corresponding to rationally dependent frequency vectors – so-called resonant tori – are generically destroyed by arbitrarily small perturbations.

In contrast, strongly non-resonant tori survive under sufficiently small perturbations. That is, there is a symplectomorphism on a neighbourhood of the torus taking the perturbed trajectory to the linear flow on a torus with unchanged frequency vector. The precise conditions are stated in the following theorem:

Theorem 2.6.2 (KAM). *Let $H(\varphi, y) = h(y)$ be an analytic function on $\mathbb{T}^n \times B^n$ with frequency map*

$$\omega(y) := \frac{\partial}{\partial y} h(y).$$

If $y_0 \in B^n$ has Diophantine frequency vector $\omega := \omega(y_0)$ and if the non-degeneracy condition *holds:*

$$\det \frac{\partial}{\partial y} \omega(y_0) \neq 0,$$

then the torus $\mathbb{T}^n \times \{y_0\}$ persists under sufficiently small perturbations of H. That is, if P is any analytic function on $\mathbb{T}^n \times B^n$ and $\epsilon > 0$ sufficiently small, the perturbed system

$$H_\epsilon = H + \epsilon P$$

admits an invariant torus $\mathcal{T}$ close to $\mathbb{T}^n \times \{y_0\}$.

Moreover, the flow γ^t of the perturbed system on $\mathcal{T}$ is conjugated via a diffeomorphism $\psi : \mathbb{T}^N \to \mathcal{T}$ to the linear flow with frequency vector ω on $\mathbb{T}^n$, i.e.

$$\psi^{-1} \circ \gamma^t \circ \psi(\varphi_0) = \varphi_0 + \omega t.$$

The basis is Kolmogorov's theorem, which we state below and whose proof is the heart of KAM theory. It tells us that we can "correct" a sufficiently small perturbation of a certain type of Hamiltonian via a symplectomorphism close to the identity. The KAM Theorem follows as an easy corollary by applying Kolmogorov's theorem to the Hamiltonian H of the system under consideration, taking into account that the conditions for ω stated in the KAM theorem imply that H is of *non-degenerate Kolmogorov form*, which we define below:

Definition 2.6.3 (Kolmogorov normal form). Let $H : \mathbb{T}^n \times B^n \to \mathbb{R}$ be an analytic function. We say that H is in **Kolmogorov normal form** (with frequency vector ω) if it is of the form

$$H(\varphi, y) = E + \omega \cdot y + Q(\varphi, y) \tag{2.13}$$

for some $E \in \mathbb{R}$, $\omega \in \mathbb{R}^n$ and a function $Q : \mathbb{T}^n \times B^n \to \mathbb{R}$ which is quadratic in y, meaning that

$$Q(\varphi, 0) = 0, \quad \frac{\partial}{\partial y} Q(\varphi, 0) = 0.$$

The Kolmogorov normal form is called **non-degenerate** if

$$\frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \det \partial_y^2 Q(\varphi, 0) d\varphi =: \langle \det \partial_y^2 Q(\cdot, 0) \rangle \neq 0.$$

Theorem 2.6.4 (Kolmogorov). *Let $H : \mathbb{T}^n \times B_r^n \to \mathbb{R}$ be in non-degenerate Kolmogorov normal form* (2.13) *with Diophantine frequency vector ω. Let $P : \mathbb{T}^n \times B_r^n \to \mathbb{R}$ be an analytic function, the "perturbation". Then there exists ϵ_0 such that for all $0 < \epsilon < \epsilon_0$ there exists a symplectomorphism*

$$\psi : \mathbb{T}^n \times B_{r_*}^n \to \mathbb{T}^n \times B_r^n$$

for some $0 < r_* < r$ which transforms the perturbed Hamiltonian $H_\epsilon := H + \epsilon P$ into Kolmogorov normal form:

$$(H_\epsilon \circ \psi)(\varphi, y) = E_* + \omega \cdot y + Q_*(\varphi, y)$$

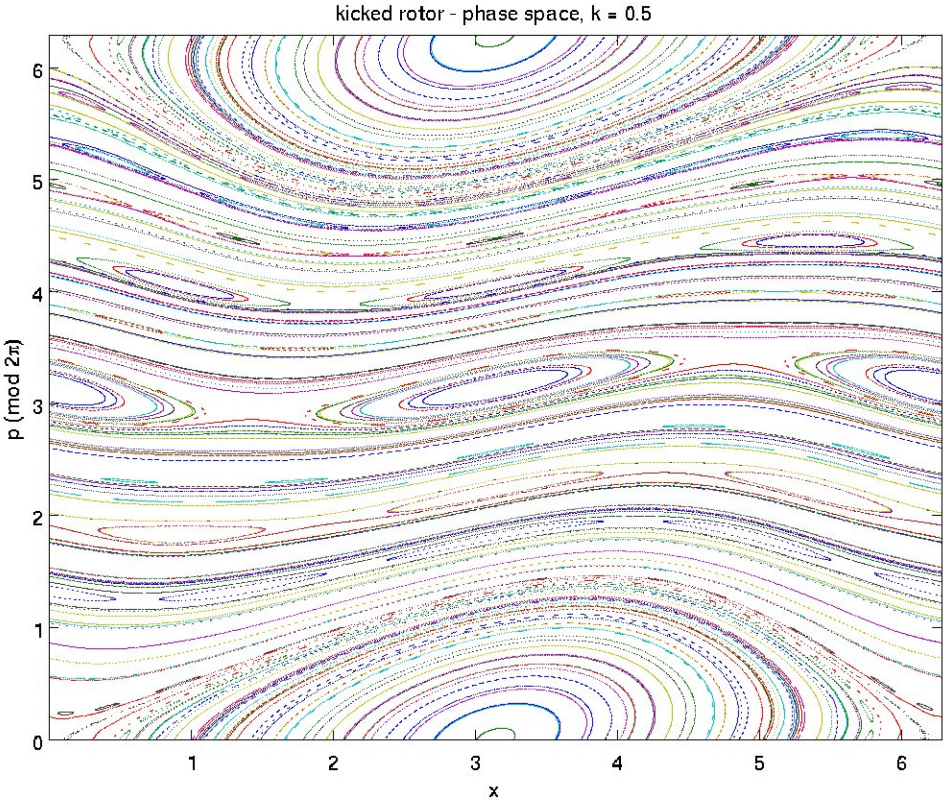

Figure 2.1: Illustration of perturbed Liouville tori using the example of the kicked rotor: The unperturbed system is the mathematical pendulum with Hamiltonian $H(x,p) = p^2/2$, $x \in S^1$ is the angle coordinate and $p \in \mathbb{R}$ is the conjugated momentum. The unperturbed system has constant momentum and the angle evolves linearly with time. If a vertical "kick" is added at every time interval then the evolution for discrete time $n \in \mathbb{N}$ can be computed to be $p_{n+1} = p_n - k \sin x_{n+1}$, $x_{n+1} = x_n + p_n$, where x_n, p_n are the values of the coordinates at time $t = n \in \mathbb{N}$. The parameter k measures the size of the perturbation. This system is illustrated above using a `matlab` program.

where[5] $\|\psi - \mathrm{id}\|, |E_* - E|$ *and* $\|Q - Q_*\|$ *are of order* ϵ.

The proof of the above theorem, which dates back to Kolmogorov [K], uses an iterative method in which the size of the perturbation is reduced quadratically in each step. The technical obstacle that has to be overcome is the famous problem of "small divisors", that is, the appearance of terms $\omega \cdot k, k \in \mathbb{Z}$ in the denominator of the Fourier series coefficients of the new perturbation. This is why the Diophantine condition is crucial, because it allows to control the size of these denominators and hence can be used to obtain estimates for the size of the new perturbation. The details of the proof are lengthy and technical and can be found e.g. in [F12].

Remark. Note that in the "new" Kolmogorov normal form K_ϵ, the constant and quadratic parts E_* resp. Q_* are slightly different from the original E resp. Q, but the frequency vector ω is the same. A concrete choice of ϵ_0 in Theorem 2.6.4 can be given, but it is not important for the purpose of this text.

[5] For functions on $\mathbb{T}^n \times B^n$ the norm $\|\cdot\|$ denotes the supremum norm; for maps $\mathbb{T}^n \times B^n \to \mathbb{T}^n \times B^n$ we regard the target space as a subset of $\mathbb{R}^{2n} \times \mathbb{R}^n$ and use the supremum norm with respect to the Euclidean norm.

Chapter 3

Integrable systems on symplectic and Poisson manifolds

In this chapter we review some well-known results about integrable systems on symplectic manifolds and their recently studied generalization to Poisson manifolds [LMV]. Moreover, we introduce the respective definitions in the b-symplectic setting [KMS, KMb]. We motivate the study of integrable systems on symplectic manifolds with some classical examples coming from physics.

In the following chapters we will study the dynamics of b-integrable systems, both commutative and non-commutative ones, in detail and in particular prove an action-angle coordinate theorem (Chapter 4) and a KAM type result (Chapter 7).

3.1 Integrable systems on symplectic manifolds

Let (M^{2n}, ω) be a symplectic manifold. An **integrable system** is given by n functions $f_1, \ldots, f_n$ which are

- in involution with respect to the Poisson bracket associated to the symplectic form ω (Equation (2.1)):

$$\{f_i, f_j\} = 0$$

- and which are functionally independent on a dense open subset of M.

The expression *integrable* has its origin in the study of Hamiltonian systems: Given a function H, the so-called Hamiltonian function, we consider the system of differential equations associated to the flow of X_H, the Hamiltonian vector field of H. These are the equations of motion of the Hamiltonian system given by H. The system is called integrable if there is an integrable system $f_1, \dots, f_n$ such that H is one of the functions f_i. As already mentioned in the Introduction, integrability of the system in the sense defined above is related to actual integration of the equations of motion by quadratures [L].

As a motivation we present the well-known example of the two-body problem and explain how the existence of integrals allows us to solve the equations of motion. The details can be found in [MHO]; we will return to this example later on and discuss some classical transformations that are used to solve the problem explicitly (see Section 8.1).

3.1.1 The two-body problem

The two-body problem is the system consisting of two bodies with masses m_1, m_2 and positions $q_1, q_2 \in \mathbb{R}^3$ moving under their mutual gravitational attraction. According to Newton's law of gravity the equations of motion are

$$m_i \ddot{q}_i = \mathcal{G} m_1 m_2 \frac{q_j - q_i}{\|q_2 - q_1\|^3} = \frac{\partial U}{\partial q_i}, \quad i, j = 1, 2, \quad i \neq j,$$

where $\mathcal{G}$ is the gravitational constant and we have introduced the negative gravitational potential

$$U := m_1 m_2 \frac{\mathcal{G}}{\|q_2 - q_1\|}.$$

We want to describe the equations of motion via the Hamiltonian formalism. The Hamiltonian function corresponds to the "energy" of the system and is obtained as the sum of kinetic and potential energy:

$$H(q_1, q_2, p_1, p_2) := E_{kin} - U = \frac{\|p_1\|^2}{2m_1} + \frac{\|p_2\|^2}{2m_2} - U,$$

where $p_i = m_i \dot{q}_i$ are the momenta. Then the evolution of the system is given by the Hamiltonian system of equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} = \mathcal{G} m_1 m_2 \frac{q_j - q_i}{\|q_2 - q_1\|^3}.$$

Here the underlying symplectic structure is the canonical one,

$$\omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2.$$

Center of mass coordinates. From the above equations of motion we see that

$$\dot{p}_1 + \dot{p}_2 = 0,$$

i.e. the total linear momentum $W := p_1 + p_2$ is preserved. Therefore the center of mass moves with constant velocity and we only have to solve the equations of motion for the relative position $w := q_2 - q_1$ of the two bodies. By introducing an appropriate set of coordinates we can transform the two-body problem to the problem of one body moving in a central force field (Kepler problem): Following [MHO] let

$$g = \nu_1 q_1 + \nu_2 q_2, \qquad G = p_1 + p_2, \tag{3.1}$$

$$w = q_2 - q_1, \qquad W = -\nu_2 p_1 + \nu_1 p_2, \tag{3.2}$$

where $\nu_i = m_i/(m_1+m_2)$. Note that g is the center of mass and G is the total linear momentum. The coordinate w is the relative position of the second body with respect to the first one. The other "momentum" coordinate W is chosen in such a way that the change of coordinates is *canonical* (i.e., the symplectic form is preserved). The coordinates (g, w, G, W) are called Jacobi coordinates.

In these coordinates the Hamiltonian is

$$H(g, w, G, W) = \frac{\|G\|^2}{2\nu} + \frac{\|W\|^2}{2M} - \mathcal{G}\frac{m_1 m_2}{\|w\|}$$

where $\nu = m_1 + m_2$ and $M = m_1 m_2/(m_1 + m_2)$.

Writing down the Hamiltonian equations explicitly

$$\dot{g} = \frac{\partial H}{\partial G} = \frac{G}{\nu}, \qquad \dot{G} = -\frac{\partial H}{\partial g} = 0,$$

$$\dot{w} = \frac{\partial H}{\partial W} = \frac{W}{M}, \qquad \dot{W} = -\frac{\partial H}{\partial w} = -\frac{m_1 m_2 w}{\|w\|^3},$$

we see that total linear momentum G is preserved and that the center of mass moves with constant velocity $\frac{G}{\nu}$. Hence the problem reduces to the second line of equations.

Physically this means that we are viewing the system from the perspective of one body with coordinates w under the influence of the central force field of a body with mass M. The upshot is that we are dealing with a Hamiltonian system on $(\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^3$ with Hamiltonian function

$$H(w, W) = \frac{\|W\|^2}{2M} - \mathcal{G}\frac{m_1 m_2}{\|w\|}.$$

This is known as the **Kepler problem**.

Planar Kepler problem. In addition to the total linear momentum, the total angular momentum is preserved. In center of mass coordinates (w, W) this means that

$$J := w \times W$$

is constant. Computing

$$\frac{d}{dt}\frac{w}{\|w\|} = \frac{(w \times \dot{w}) \times w}{\|w\|^3} = \frac{J \times w}{\|w\|^3} \tag{3.3}$$

we see that for $J = 0$ the motion is collinear. We consider the case $J \neq 0$: Both w and $W = \dot{w}$ lie in the plane perpendicular to J. Therefore the problem reduces to two degrees of freedom, the so-called *planar Kepler problem*.

After these simplifications the problem can be solved explicitly: Integrating Equation (3.3) yields after some computations (see [MHO]) the equation of a conic section. Therefore, depending on the eccentricity, the body moves on a parabola, an ellipse or a hyperbola.

In summary, the 6-degree-of-freedom system given by the two-body problem has six independent integrals given by the components of total linear and angular momentum. (The energy is an integral too, but depends on these integrals.)

3.1.2 Other classical examples of integrable systems

We provide a list of other well-known examples of integrable systems:

Example 3.1.1 (Integrable Hamiltonian systems).

(a) Any 2-dimensional Hamiltonian system with $dH \neq 0$ (a.e.) is integrable. For instance, this includes the mathematical pendulum.

(b) N bodies in a central force field: As we have seen above, the two-body problem can be reduced to the problem of a single body in a central force field via appropriate changes of coordinates. More generally, we can consider the problem of $N \geq 1$ bodies under the influence of a central force, e.g. the motion of planets in the gravitational field of the sun. If we neglect any interaction between the bodies, this is an integrable system as well. In contrast, the classical N body problem, where mutual interaction between all bodies is allowed, is not integrable for $N \geq 3$, as was shown in 1887 by H. Bruns.

(c) A rigid body fixed at its centre of gravity in a constant gravitational field: The system of a rigid body fixed at a point has configuration space $SO(3)$. Therefore, in addition to energy, one more constant of motion is needed to obtain integrability. If the fixed point is the centre of gravity, then such an integral is given by $\|M\|$, the norm of the total angular momentum. Instead of assuming that the fixed point is the centre of gravity, other (non-trivial) conditions can be given which guarantee integrability. A list of these cases is given in [Au].

(d) The spherical pendulum: Consider a pendulum in three-dimensional space, where a particle of unit mass is fixed to a rod of unit length and moves under the influence of gravity. The configuration space is S^2, in particular we have two degrees of freedom. Using spherical coordinates (θ, φ), where θ is the polar angle and φ the azimuthal angle, the Hamiltonian H, which as usual is kinetic plus potential energy, takes the form
$$H(\varphi, \theta, p_\varphi, p_\theta) = \frac{1}{2}\left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}\right) + \cos\theta.$$
where $p_\varphi = \sin^2\theta\,\dot{\varphi}$ and $p_\theta = \dot{\theta}$ are the conjugate momenta. Since H is independent of φ, the conjugate momentum p_φ is conserved and yields another integral. Therefore, the system is integrable.

3.1.3 Action-angle coordinates for integrable systems on symplectic manifolds

The existence of an integrable system has profound implications on the dynamics of the system. In a neighbourhood of a compact level set ("Liouville torus") of the integrals, the manifold is a fibration of tori and the motion is linear on these tori. This is the content of the action-angle-coordinate theorem:

Theorem 3.1.2. (Liouville-Mineur-Arnold Theorem)
Let (M^{2n}, ω) be a symplectic manifold. Let $F = (f_1, \ldots, f_n)$ be an integrable system on M. Assume that m is a regular point[1] *of F and that the level set of F through m, which we denote by $\mathcal{F}_m$, is compact and connected.*

Then $\mathcal{F}_m$ is a torus and on a neighborhood U of $\mathcal{F}_m$ there exist $\mathbb{R}$-valued smooth functions
$$(p_1, \ldots, p_n)$$

[1] i.e. the differentials df_i are independent at m

and $\mathbb{R}/\mathbb{Z}$*-valued smooth functions*

$$(\theta_1, \dots, \theta_n)$$

such that:

1. *The functions* $(\theta_1, \dots, \theta_n, p_1, \dots, p_n)$ *define a diffeomorphism* $U \simeq \mathbb{T}^n \times B^n$*.*
2. *The symplectic structure can be written in terms of these coordinates as*
$$\omega = \sum_{i=1}^{n} d\theta_i \wedge dp_i.$$
3. *The leaves of the surjective submersion* $F = (f_1, \dots, f_s)$ *are given by the projection onto the second component* $\mathbb{T}^n \times B^n$*, in particular, the functions* $f_1, \dots, f_s$ *depend only on* $p_1, \dots, p_n$*.*

The coordinates p_i *are called action coordinates; the coordinates* θ_i *are called angle coordinates.*

Remark. In physics, usually one of the integrals f_i of Theorem 3.1.2 is the energy H, e.g. $f_1 = H$, and motion is given by the flow of the Hamiltonian vector field of H. Statement (3) in Theorem 3.1.2 implies that H is constant along the level sets of the functions f_i. Moreover, since

$$df_i(X_H) = \{H, f_i\} = 0,$$

the vector field X_H is tangent to the level sets. More precisely, in the action-angle coordinate chart, the flow of X_H is linear on the invariant tori.

We refer to [F13] for a very interesting survey about the action-angle coordinate theorem with more information about its historical development.

3.1.4 Non-commutative integrable systems

For some Hamiltonian systems the motion takes place on smaller tori, see also [BJ]. In this case there are more independent integrals than half the dimension of the phase space. In physics such a system is called "superintegrable", here we use the term non-commutative:

Definition 3.1.3 (Non-commutative integrable system). A non-commutative integrable system of rank r on a $2n$-dimensional symplectic manifold (M, ω) is an s-tuple of functions

$$F = (f_1, \dots, f_r, f_{r+1}, \dots, f_s)$$

such that the following conditions are satisfied:

(1) The differentials $df_1, \dots, df_s$ are linearly independent on a dense open subset of M;

(2) The functions $f_1, \dots, f_r$ are in involution with the functions $f_1, \dots, f_s$;

(3) $r + s = 2n$.

For non-commutative integrable systems an action-angle coordinate theorem holds in analogy to the commutative case. We will not state the theorem here, since it is a special case of the action-angle coordinate theorem for non-commutative systems on Poisson manifolds, presented below.

3.2 Integrable systems on Poisson manifolds

The concept of integrable systems on Poisson manifolds was studied in [LMV], both in the commutative and non-commutative case. Here we review the definitions and results for the commutative case, which we will adapt later on (Section 3.3) to the particular case of b-Poisson manifolds. Subsequently we will consider non-commutative systems.

Definition 3.2.1 (Integrable system on a Poisson manifold). Let (M, Π) be a Poisson manifold of maximal rank $2r$. An s-tuple of functions $F = (f_1, \dots, f_s)$ on M is called an **integrable system** on (M, Π) if

(1) $f_1, \dots, f_s$ are in involution, i.e. $\{f_i, f_j\} = 0$ for all $i, j = 1, \dots, s$;

(2) $df_1 \wedge \dots \wedge df_s \in \Lambda^s T^*(M)$ is non-zero on a dense subset of M;

(3) $r + s = \dim M$.

Viewed as a map, $F : M \to \mathbb{R}^s$ is called the **momentum map** of (M, Π, F).

Remark. The second condition means that $f_1, \dots, f_s$ are functionally independent on a dense set.

Example 3.2.2 (A generic example). Consider the manifold $M = \mathbb{T}^r \times \mathbb{R}^s$ with coordinates

$$(\theta_1, \dots, \theta_r, p_1, \dots, p_s)$$

equipped with the Poisson structure

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial p_i}$$

Then the functions $(p_1, \dots, p_s)$ define an integrable system on (M, Π).

As we will see in Theorem 3.2.4 below, any integrable system semilocally takes this form, more precisely in the neighborhood of a regular compact connected level set of its integrals $(f_1, \dots, f_s)$.

Liouville tori. Let (M, Π, F) be an integrable system where the maximal rank of Π is $2r$. We denote the non-empty open subset of M where Π has rank $2r$ by M_r and the non-empty open set where the differentials $df_1, \dots, df_s$ are independent by $\mathcal{U}_F$.

Proposition 3.2.3. *On the intersection $M_r \cap \mathcal{U}_F$ of M, the Hamiltonian vector fields $X_{f_1}, \dots, X_{f_s}$ define an involutive distribution of rank r.*

Proof. The kernel of the anchor map at a point in $M_r \cap \mathcal{U}_F$ has dimension

$$\dim M - 2r = s - r.$$

Therefore, the vector space spanned by $X_{f_1}, \dots, X_{f_s}$ must be at least r-dimensional. On the other hand, the commutativity of $f_1, \dots, f_s$ means that $df_i(X_{f_j}) = 0$ and therefore $X_{f_1}, \dots, X_{f_s}$ lie in the level set of $f_1, \dots, f_s$, hence they can only span a vector space of dimension at most $\dim M - s = r$. Therefore, we conclude that the distribution defined by $X_{f_1}, \dots, X_{f_s}$ on $M_r \cap \mathcal{U}_F$ is r-dimensional.

This distribution is involutive since

$$[X_{f_i}, X_{f_j}] = X_{\{f_i, f_j\}} = 0.$$

□

We denote the foliation corresponding to the integrable distribution given by $X_{f_1}, \dots, X_{f_s}$ on $M_r \cap \mathcal{U}_F$ by $\mathcal{F}$. Its leaves are r-dimensional.

Let $\mathcal{F}_m$ be the leaf passing through a point $m \in M_r \cap \mathcal{U}_F$. We will only deal with the case where $\mathcal{F}_m$ is compact. Under this assumption, $\mathcal{F}_m$ is a compact r-dimensional manifold, equipped with r independent commuting vector fields, hence it is diffeomorphic to an r-dimensional torus $\mathbb{T}^r$. The torus $\mathcal{F}_m$ is called a *Liouville torus* of F.

3.2.1 Action-angle coordinates for integrable systems on Poisson manifolds

The action-angle coordinate theorem proved in [LMV] gives a semilocal description of the Poisson structure around a Liouville torus of an integrable system:

Theorem 3.2.4. *Let (M, Π) be a Poisson manifold of dimension n and let $2r$ be the maximal rank of Π. Let $F = (f_1, \dots, f_s)$ be an integrable system on M and suppose that $\mathcal{F}_m$ is a Liouville torus, where $m \in M_{F,r} \cap \mathcal{U}_F$. Then there exist $\mathbb{R}$-valued smooth functions*

$$(p_1, \dots, p_s)$$

and $\mathbb{R}/\mathbb{Z}$-valued smooth functions

$$(\theta_1, \dots, \theta_r),$$

defined in a neighbourhood U of $\mathcal{F}_m$ such that

1. *The functions $(\theta_1, \dots, \theta_r, p_1, \dots, p_s)$ define a diffeomorphism $U \simeq \mathbb{T}^r \times B^s$;*

2. *The Poisson structure can be written in terms of these coordinates as,*

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial p_i};$$

3. *The leaves of the surjective submersion $F = (f_1, \dots, f_s)$ are given by the projection onto the second component $\mathbb{T}^r \times B^s$. In particular, the functions $f_1, \dots, f_s$ depend on $p_1, \dots, p_s$ only.*

The functions $\theta_1, \dots, \theta_r$ are called angle coordinates*, the functions $p_1, \dots, p_r$ are called* action coordinates *and the remaining coordinates $a_{r+1}, \dots, a_s$ are called* transverse coordinates*.*

3.3 b-integrable systems

For b-symplectic manifolds we introduce a new definition of "integrable system" where we allow the integrals to be b-functions [KMS]. Such a "b-integrable system" on a $2n$-dimensional manifold consists of n integrals, just as in the symplectic case.

Since b-symplectic (b-Poisson) manifolds are a special case of Poisson manifolds, a first thought would be to use the standard definition of integrable systems for Poisson manifolds (Definition 3.2.1). However, on the critical hypersurface of the b-symplectic manifolds, such a system will only define a distribution of rank at most $2n-2$. This is why the need to consider b-functions arises: in this way we obtain a distribution of (b-)Hamiltonian vector fields that has rank n on Z.[2]

The theorem we aim at, which is the natural generalization of the action-angle coordinate theorem in the *symplectic* case, will semilocally put the b-symplectic structure into the form

$$\omega = \frac{c}{t} d\theta_1 \wedge dt + \sum_{i=2}^{n} d\theta_i \wedge dp_i, \tag{3.4}$$

where θ_i are S^1 valued coordinates and the "action coordinates" $t, p_2, \ldots, p_n$ depend only on the integrals of the system. Here, t is a defining function of Z and the number $c \in \mathbb{R}$ is the modular period of the connected component of Z in which the Liouville torus lies (see Definition 2.2.14).

We see that the vector field $\frac{\partial}{\partial \theta_1}$ in the expression (3.4) is actually not a Hamiltonian vector field of the action coordinate t (which is a defining function of Z), but it *is* a b-Hamiltonian vector field of $\log |t|$. This again motivates the need to consider b-functions as integrals:

Definition 3.3.1 (**b-integrable system**). A b-integrable system on a $2n$-dimensional b-symplectic manifold (M^{2n}, ω) is a set of b-functions $F = (f_1, \ldots, f_n)$ satisfying,

- the functions are pairwise commuting, $\{f_i, f_j\} = 0$ for all i, j;
- $df_1 \wedge \cdots \wedge df_n$ is nonzero as a section of $\Lambda^n({}^bT^*(M))$ on a dense subset of M and on a dense subset of Z.

We call points in M where the second condition holds **regular** points.

[2] More precisely the rank is n only for regular points, see Definition 3.3.1

Remark. Note that $df_1 \wedge \cdots \wedge df_n$ is nonzero at a point as an element of $\Lambda^n({}^bT^*M)$ if and only if the vector fields $X_{f_1}, \ldots, X_{f_n}$ are linearly independent at that point, since the map ${}^bTM \to {}^bT^*M$ (Equation (2.9)) induced by ω is bijective. Moreover the condition implies that at least one of the f_i most be non-smooth, i.e. a genuine b-function.

Liouville tori. On the set of regular points, the distribution given by $X_{f_1}, \ldots, X_{f_n}$ defines a foliation $\mathcal{F}$ with n-dimensional leaves. We denote the integral manifold through a regular point $m \in M$ by $\mathcal{F}_m$. As before, if the integral manifold $\mathcal{F}_m$ is compact, then it is an n-dimensional torus called **Liouville torus**. Because the X_{f_i} are b-vector fields and are therefore tangent to Z, any Liouville torus that intersects Z actually lies inside Z.

3.3.1 Equivalent b-integrable systems.

We write a b-integrable system as a triple (M, ω, F) where M is a manifold, ω a b-symplectic form and F the set of integrals.

Definition 3.3.2 (Equivalence of b-integrable systems). We say that two b-integrable systems (M_1, ω_1, F_1) and (M_2, ω_2, F_2) are **equivalent** if there exists a Poisson diffeomorphism ψ and a map $\varphi : \mathbb{R}^s \to \mathbb{R}^s$ such that the following diagram commutes:

$$\begin{tikzcd} (M_1, \omega_1) \arrow[r, "\psi"] \arrow[dr, "F_1"'] & (M_2, \omega_2) \arrow[d, "\varphi \circ F_2"] \\ & \mathbb{R}^s \end{tikzcd}$$

We will not distinguish between equivalent integrable systems, since the action-angle coordinate theorem will automatically hold for all equivalent systems too.

The notion of equivalent systems allows us to think about a simple "normal" form. The following proposition is a first result in this direction. It is a consequence of Proposition 3.5.3 for non-commutative b-integrable systems, which we will prove in Section 3.5:

Proposition 3.3.3. *Near a regular point of Z, any b-integrable system on a b-symplectic manifold is equivalent to a b-integrable system of the form $F = (f_1, \ldots, f_n)$, where f_1 is a b-function and $f_2, \ldots, f_n$ are C^∞ functions.*

In fact, we may always assume that $f_1 = \log|t|$, *where* t *is a global defining function for* Z.

3.3.2 Action-angle coordinates for b-integrable systems

We have now introduced the background to state the action-angle coordinate theorem for b-integrable systems, which was first proved in [KMS]. The proof will be carried out in Chapter 4 of this thesis and we will use the theorem later on to prove a KAM-type stability result for b-integrable systems (Chapter 7).

Theorem A (Action-angle coordinates for b-integrable systems)**.** *Let* (M, ω) *be a* b*-symplectic manifold with critical hypersurface* Z. *Let* F *be a* b*-integrable system on* (M, ω) *and let* $m \in Z$ *be a regular point of the system lying inside the critical hypersurface. Assume that the integral manifold* $\mathcal{F}_m$ *containing* m *is compact, i.e. a Liouville torus. Then there exists an open neighbourhood* U *of the torus* $\mathcal{F}_m$ *and a diffeomorphism*

$$(\theta_1, \ldots, \theta_n, t, a_2, \ldots, a_n) : U \to \mathbb{T}^n \times B^n,$$

where t *is a defining function for* Z, *such that*

$$\omega|_U = \frac{c}{t} d\theta_1 \wedge dt + \sum_{i=2}^{n} d\theta_i \wedge dp_i.$$

Moreover, the functions $t, p_2, \ldots, p_n$ *depend only on* F. *The number* c *is the modular period of the component of* Z *containing* m.

The S^1*-valued functions*

$$\theta_1, \ldots, \theta_r$$

are called angle coordinates *and the* $\mathbb{R}$*-valued functions*

$$t, p_2, \ldots, p_r$$

are called action coordinates.

3.4 Non-commutative integrable systems on Poisson manifolds

We have already defined non-commutative integrable systems in the symplectic case (Definition 3.1.3). The natural generalization of this concept to Poisson manifolds is the following:

Definition 3.4.1 (Non-commutative integrable system on a Poisson manifold). Let (M, Π) be a Poisson manifold. A **non-commutative integrable system** of rank r on M is an s-tuple of functions

$$F = (f_1, \dots, f_r, f_{r+1}, \dots, f_s)$$

such that

(1) $f_1, \dots, f_s$ are independent (i.e. their differentials are independent on a dense open subset of M);

(2) The functions $f_1, \dots, f_r$ are in involution with the functions $f_1, \dots, f_s$;

(3) $r + s = \dim M$;

(4) The Hamiltonian vector fields of the functions $f_1, \dots, f_r$ are linearly independent at some point of M.

Viewed as a map, $F : M \to \mathbb{R}^s$ is called the **momentum map** of (M, Π, F).

We call the first r functions $(f_1, \dots, f_r)$ the commuting part of the system and the last $s - r$ functions the non-commuting part.

When all the integrals commute, i.e. $r = s$, then we are dealing with the case of a commutative integrable system discussed in Section 3.2.

Remark. Unlike the commutative case, we have to explicitly demand the independence of the Hamiltonian vector fields of $f_1, \dots, f_r$ (condition (4) above). For instance, on $\mathbb{R}^{3r}$ endowed with coordinates

$$(x_1, \dots, x_{2r}, z_1, \dots, z_r)$$

and Poisson structure

$$\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4} + \dots + \frac{\partial}{\partial x_{2r-1}} \wedge \frac{\partial}{\partial x_{2r}}$$

the set of functions $z_1, \dots, z_r, x_1, \dots, x_r$ satisfies conditions (1)-(3) but the Hamiltonian vector fields of $z_1, \dots, z_r$ are zero. In the action-angle coordinate theorem, we explicitly restrict our attention to points where the Hamiltonian vector fields of the commuting part of the system are independent.

An example, and at the same time the semilocal normal form of any non-commutative integrable system on a Poisson manifold, is the following system:

Example 3.4.2 (Generic example). Consider the manifold $\mathbb{T}^r \times \mathbb{R}^s$ with standard coordinates

$$(\theta_1, \dots, \theta_r)$$

on $\mathbb{T}^r$ and standard coordinates

$$(p_1, \dots, p_r, z_1, \dots, z_{s-r})$$

on $\mathbb{R}^s$. We endow $\mathbb{T}^r \times \mathbb{R}^s = \mathbb{T}^r \times \mathbb{R}^r \times \mathbb{R}^{s-r}$ with the Poisson structure

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial a_i} + \pi'$$

where π' is any Poisson structure on $\mathbb{R}^{s-r}$. Then the functions

$$(p_1, \dots, p_r, z_1, \dots, z_{s-r})$$

define a non-commutative integrable system of rank r.

Liouville tori. Let (M, Π, F) be a non-commutative integrable system of rank r. We denote the non-empty subset of M where the differentials $df_1, \dots, df_s$ (resp. the Hamiltonian vector fields $X_{f_1}, \dots, X_{f_r}$) are independent by $\mathcal{U}_F$ (resp. $M_{F,r}$).

On the non-empty open subset $M_{F,r} \cap \mathcal{U}_F$ of M, the Hamiltonian vector fields $X_{f_1}, \dots, X_{f_r}$ by definition generate an integrable distribution of rank r and hence a foliation $\mathcal{F}$ with r-dimensional leaves, see [LMV]. As in the commutative case, a compact leaf of this foliation is an r-dimensional torus called Liouville torus.

3.4.1 Action-angle coordinates for non-commutative integrable systems on Poisson manifolds

The action-angle coordinate theorem proved in [LMV] (see Theorem 1.1 therein) gives a semilocal description of the Poisson structure around a Liouville torus of a non-commutative integrable system:

Theorem 3.4.3. *Let (M, Π, F) be a non-commutative integrable system of rank r, where $F = (f_1, \dots, f_s)$ and suppose that $\mathcal{F}_m$ is a Liouville torus, where $m \in M_{F,r} \cap \mathcal{U}_F$. Then there exist $\mathbb{R}$-valued smooth functions*

$$(p_1, \dots, p_r, z_1, \dots, z_{s-r})$$

and $\mathbb{R}/\mathbb{Z}$-valued smooth functions

$$(\theta_1, \ldots, \theta_r),$$

defined in a neighborhood U of $\mathcal{F}_m$, and functions $\phi_{kl} = -\phi_{lk}$, which are independent of $\theta_1, \ldots, \theta_r, p_1, \ldots, p_r$, such that

1. *The functions $(\theta_1, \ldots, \theta_r, p_1, \ldots, p_r, z_1, \ldots, z_{s-r})$ define a diffeomorphism $U \simeq \mathbb{T}^r \times B^s$;*

2. *The Poisson structure can be written in terms of these coordinates as,*

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial p_i} + \sum_{k,l=1}^{s-r} \phi_{kl}(z) \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_l};$$

3. *The leaves of the surjective submersion $F = (f_1, \ldots, f_s)$ are given by the projection onto the second component $\mathbb{T}^r \times B^s$, in particular, the functions $f_1, \ldots, f_s$ depend on $p_1, \ldots, p_r, z_1, \ldots, z_{s-r}$ only.*

The functions $\theta_1, \ldots, \theta_r$ are called angle coordinates, *the functions $p_1, \ldots, p_r$ are called* action coordinates *and the remaining coordinates $z_1, \ldots, z_{s-r}$ are called* transverse coordinates.

3.4.2 Casimir-basic functions

In the case of non-commutative integrable systems, the target space of the functions $F = (f_1, \ldots, f_s) : M \to \mathbb{R}^s$ inherits a Poisson structure which is non-trivial. We denote the target space by $V := F(M)$. The Poisson structure on V is defined in the following way [LMV]:

$$\{g, h\}_V \circ F = \{g \circ F, h \circ F\}, \tag{3.5}$$

where g, h are functions on V and on the right hand side the bracket $\{\, , \}$ is the Poisson bracket on M. We denote the bivector field associated with $\{\, , \}_V$ by Π_V.

Remark. The bracket $\{\cdot, \cdot\}_V$ is completely determined by the brackets of the integrals $\{f_i, f_j\}$, since this corresponds to evaluating the left-hand side of Equation (3.5) on the coordinate functions of V. In particular, a commuting integrable system induces the zero Poisson structure on the target space since $\{f_i, f_j\} = 0$ for all i, j.

An F-basic function on M is a function of the form $h \circ F$ where h is a function on V. Equivalently, a function is F-basic if and only if it is constant on all the level sets of F. Since these level sets are spanned by $X_{f_1}, \ldots, X_{f_r}$, we can characterize an F-basic functions g by the property that $X_{f_i}(g) = 0$ for $i = 1, \ldots, r$.

The Poisson bracket on V allows us to distinguish a particular class of F-basic functions [LMV], which will play an important role in the proof of the action-angle coordinate theorem:

Definition 3.4.4. A smooth function g on M is said to be a **Casimir-basic function**, or **Cas-basic function** for short, if there exists a Casimir function h on (V, Π_V) such that $g = h \circ F$.

Recall from Definition 2.1.2 that a Casimir function for a Poisson structure is one whose Hamiltonian vector field is zero, in other words a Casimir function commutes with all other functions.

We state the following useful characterisation of Cas-basic functions proved in [LMV]:

Proposition 3.4.5. *A function is Cas-basic if and only if it commutes with all F-basic functions.*

Proof. Suppose that $g \in C^\infty(M)$ is Cas-basic, $g = h \circ F$. Then clearly for an F-basic function $k \circ F$ we have

$$0 = \{h, k\}_V \circ F = \{g, k \circ F\}.$$

To show the other direction, let $g \in C^\infty(M)$ be in involution with all F-basic functions. Then $X_{f_i}(g) = \{g, f_i\} = 0$ for $i = 1, \ldots, r$, hence g is F-basic, $g = h \circ F$ for some function h on B^s. If $k \in C^\infty(B^s)$, then $k \circ F$ is constant on the fibres of F, so that

$$\{h, k\}_B \circ F = \{g, k \circ F\} = 0,$$

where we have used that F is a Poisson map. It follows that $\{h, k\}_B = 0$ for all functions k on B^s, hence that g $(= h \circ F)$ is Cas-basic. This shows one implication of (1), the other one is clear. $\square$

3.5 Non-commutative b-integrable systems

As in the commutative case, to obtain a suitable concept of "non-commutative integrable system" similar to the symplectic case, we have to consider b-functions [KMb]:

Definition 3.5.1 (Non-commutative b-integrable system)*.* A non-commutative b-integrable system of rank r on a $2n$-dimensional b-symplectic manifold (M^{2n}, ω) is an s-tuple of functions

$$F = (f_1, \dots, f_r, f_{r+1}, \dots, f_s)$$

where $f_1, \dots, f_r$ are b-functions and $f_{r+1}, \dots, f_s$ are smooth such that the following conditions are satisfied:

(1) The differentials $df_1, \dots, df_s$ are linearly independent as b-cotangent vectors on a dense open subset of M and on a dense open subset of Z;

(2) The functions $f_1, \dots, f_r$ are in involution with the functions $f_1, \dots, f_s$;

(3) $r + s = 2n$;

(4) The Hamiltonian vector fields of the functions $f_1, \dots, f_r$ are linearly independent as smooth vector fields at some point of Z.

We call the first r functions $(f_1, \dots, f_r)$ the commuting part of the system and the last $s - r$ functions the non-commuting part.

For $r = s = n$ we are dealing with the commutative case, which we already discussed above.

Liouville tori. We denote the non-empty subsets of M where condition (1) resp. (4) is satisfied by $\mathcal{U}_F$ resp. $M_{F,r}$. The points of the intersection $M_{F,r} \cap \mathcal{U}_F$ are called *regular*. As in the general Poisson case, the Hamiltonian vector $X_{f_1}, \dots, X_{f_r}$ fields define an integrable distribution of rank r on this set and we denote the corresponding foliation by $\mathcal{F}$. If the leaf through a point $m \in M$ is compact, then it is an r-torus ("**Liouville torus**"), denoted $\mathcal{F}_m$.

Remark. In the symplectic case, if the differentials df_i $(i = 1, \dots, r)$ are linearly independent at a point p, then also the corresponding Hamiltonian vector fields X_{f_i} are independent at p. However, the situation is more delicate in the b-symplectic case. The differentials df_i are b-one-forms. At a point p where the df_i are independent as b-cotangent vectors, the corresponding Hamiltonian vector fields X_{f_i} are independent at p as b-tangent vectors. However, for $p \in Z$ the natural map ${}^bTM|_p \to TZ|_p$ is not injective and therefore we cannot guarantee independence of the X_{f_i} as smooth vector fields. This is why the condition (4) is needed, which already appeared in the general Poisson case. As an example, consider $\mathbb{R}^2$ with standard coordinates (t, z) and b-symplectic structure

$$\frac{1}{t} dt \wedge dz.$$

Then the function z has a differential dz which is non-zero at all points of $\mathbb{R}^2$, but the Hamiltonian vector field of z is $t\frac{\partial}{\partial t}$ and vanishes along $Z = \{t = 0\}$ as a smooth vector field. We do not allow this kind of systems in our definition, since we are interested precisely in the dynamics on Z and the existence of r-dimensional Liouville tori there.

3.5.1 The Poisson structure on the target space

Similar to the case of non-commutative integrable systems on general Poisson manifolds, we can define a Poisson structure on the target space of a non-commutative b-integrable system F on a b-symplectic manifold (M, ω) with critical hypersurface Z. We denote the Poisson bracket associated to ω by $\{\cdot, \cdot\}$. Let $V := F(M) \cap \mathbb{R}^s$ be the "finite" target space of the integrals F. (If we want to emphasize the functions F we are referring to, we will also write V_F.)

The space V inherits a Poisson structure $\{\cdot, \cdot\}_V$ satisfying the following property on $M \backslash Z$:

$$\{g, h\}_V \circ F = \{g \circ F, h \circ F\},$$

where g, h are functions on V. Note that the values of the brackets $\{f_i, f_j\}$ on M uniquely define the Poisson bracket $\{\cdot, \cdot\}_V$.

3.5.2 Equivalent non-commutative b-integrable systems.

As in the commutative case, we use the notation (M, ω, F) for a non-commutative b-integrable system F on the b-symplectic manifold (M, ω).

Definition 3.5.2 (Equivalence of non-commutative b-integrable systems)*.* We say that two non-commutative b-integrable systems (M_1, ω_1, F_1) and (M_2, ω_2, F_2) are **equivalent** if there exists a Poisson diffeomorphism ψ and a *Poisson* map

$$\mu : V_{F_1} \to V_{F_2}$$

such that the following diagram commutes:

$$\begin{tikzcd}
(M_1,\omega_1) \arrow[r, "\psi"] \arrow[d, "F_1"'] & (M_2,\omega_2) \arrow[d, "F_2"] \\
V_{F_1} \arrow[r, "\mu"'] & V_{F_2}
\end{tikzcd}$$

Here, μ is a Poisson map with respect to the Poisson structures induced on V_{F_1} and V_{F_2} as defined above.

We will not distinguish between equivalent systems: if the action-angle coordinate theorem that we will prove holds for one system then it holds for all equivalent systems too.

We prove a first "normal form" result for non-commutative b-integrable systems:

Proposition 3.5.3. *Let (M,ω) be a b-symplectic manifold of dimension $2n$ with critical hypersurface Z. Given a non-commutative b-integrable system $F = (f_1,\dots,f_s)$ of rank r there exists an equivalent non-commutative b-integrable system of the form*

$$(\log|t|, f_2, \dots, f_s)$$

where t is a defining function of Z and the functions $f_2,\dots,f_s$ are smooth.

Proof. First, assume that one of the functions $f_1,\dots,f_r$ is a genuine b-function, without loss of generality $f_1 = g + c\log|t'|$ where $c \neq 0$ and t' is a defining function of Z. Dividing f_1 by the constant c and replacing the defining function t' by $t := e^g t'$, we can restrict to the case $f_1 = \log|t|$. We subtract an appropriate multiple of f_1 from the other functions $f_2,\dots,f_r$ so that they become smooth. Note that this does not affect their independence nor the commutativity condition for $f_1,\dots,f_r$, since f_1 commutes with all the integrals. Also, since these operations do not affect the non-commutative part of the system, the induced Poisson bracket on the target space (cf. Section 3.4.2) remains unchanged. Hence we have obtained an equivalent b-integrable system of the desired form.

We prove by contradiction that it is indeed not possible that all the functions $f_1,\dots,f_s$ are smooth. If they are, then the differentials $df_1,\dots,df_s$ are standard de Rham forms and on a point $p \in Z$, the corresponding b-cotangent vectors are $\iota^*(df_1),\dots,\iota^*(df_s)$, where we use the identification described in

Section 2.2.4 and $\iota : Z \hookrightarrow M$ is the inclusion. Since $df_1, \dots, df_s$ are independent as b-de Rham forms at a point $p \in \mathcal{U}_F \cap M_{F,r}$, we have that $\iota^*(df_1), \dots, \iota^*(df_s)$ are independent at every point in $Z \cap \mathcal{U}_F \cap M_{F,r}$. But this means that they define a codimension s-foliation on $Z \cap \mathcal{U}_F \cap M_{F,r}$, i.e. the leaves are $2n-1-s$-dimensional. On the other hand, the Hamiltonian vector fields $X_{f_1}, \dots, X_{f_r}$ are tangent to the leaves of the submersion given by F, because $f_1, \dots, f_r$ commute with all $f_j, j = 1, \dots, s$. Moreover, as Poisson vector fields they are tangent to Z. But $r = 2n - s > 2n - 1 - s$, contradiction. □

Remark. Recall that the Liouville tori of a non-commutative b-integrable system F are, by definition, the leaves of the foliation induced by $X_{f_i}, i = 1, \dots, r$ on $\mathcal{U}_F \cap M_{F,r}$. A Liouville torus that intersects Z lies inside Z, since the Hamiltonian vector fields are Poisson vector fields and therefore tangent to Z. Moreover, since at least one of the first r integrals $f_1, \dots, f_r$ has non-vanishing "log" part, the Liouville tori inside Z are *transverse* to the symplectic leaves.

Remark. In the definition of a non-commutative b-integrable system, we only allow the commuting part of the system $f_1, \dots, f_r$ to contain b-functions. This condition is used in the proof of Proposition 3.5.3. Otherwise we could not perform the linear combinations required to obtain the desired normal form without affecting the commutativity properties of the functions.

3.5.3 Action-angle coordinates for non-commutative b-integrable systems

We now state the action-angle coordinate theorem for non-commutative b-integrable systems, which we first showed in [KMb] and whose proof we will present in Chapter 5.

Theorem B. *Let (M, ω) be a b-symplectic manifold with critical hypersurface Z. Let F be a non-commutative b-integrable system on (M, ω) of rank r and let $m \in Z$ be a regular point of the system lying inside the critical hypersurface, $m \in M_{F,r} \cap \mathcal{U}_{\mathcal{F}} \cap Z$. Assume that the integral manifold $\mathcal{F}_m$ containing m is compact, i.e. a Liouville torus. Then there exists an open neighbourhood U of the torus $\mathcal{F}_m$ and a diffeomorphism*

$$(\theta_1, \dots, \theta_r, t, p_2, \dots, p_r, x_1, \dots, x_\ell, y_1 \dots, y_\ell) : U \to \mathbb{T}^r \times B^s,$$

where $\ell = n - r = \frac{s-r}{2}$ *and* t *is a defining function of* Z*, such that*

$$\omega|_U = \frac{c}{t} d\theta_1 \wedge dt + \sum_{i=2}^{r} d\theta_i \wedge dp_i + \sum_{k=1}^{\ell} dx_k \wedge dy_k.$$

Moreover, the functions $f_1, \ldots, f_s$ *depend on* $t, p_2, \ldots, p_r, x_1, \ldots, x_\ell, y_1 \ldots, y_\ell$ *only. The number* c *is the modular period of the component of* Z *containing* m*.*

The S^1*-valued functions*

$$\theta_1, \ldots, \theta_r$$

are called angle coordinates*, the* $\mathbb{R}$*-valued functions*

$$t, p_2, \ldots, p_r$$

are called action coordinates *and the remaining* $\mathbb{R}$*-valued functions*

$$x_1, \ldots, x_\ell, y_1 \ldots, y_\ell$$

are called transverse coordinates*.*

Chapter 4

Action-angle coordinates for b-integrable systems

We have already stated the action-angle coordinate theorem for b-integrable systems, Theorem A. The proof will be the main content of this chapter. It borrows, on the one hand, ingredients from the proof in the general Poisson case given in [LMV], and on the other hand uses the specific properties of b-symplectic manifolds to achieve a *stronger* result in this case, which specifically concerns the critical hypersurface. The resulting theorem is similar to the symplectic case: semilocally around a Liouville torus contained in the critical hypersurface there are action-angle coordinates such that the b-symplectic structure is the "b-Darboux form" given in Equation (3.4) and the foliation of the integrable system on this chart is a trivial torus fibration $\mathbb{T}^n \times B^n$ given by the level sets of the action coordinates. The results of this chapter were first proved in [KMS] (joint work with Eva Miranda and Geoffrey Scott).

Theorem A (Action-angle coordinates for b-integrable systems)**.** *Let (M, ω) be a b-symplectic manifold with critical hypersurface Z. Let F be a b-integrable system on (M, ω) and let $m \in Z$ be a regular point of the system lying inside the critical hypersurface. Assume that the integral manifold $\mathcal{F}_m$ containing m is compact, i.e. a Liouville torus. Then there exists an open neighbourhood U of the torus $\mathcal{F}_m$ and a diffeomorphism*

$$(\theta_1, \ldots, \theta_n, t, a_2, \ldots, a_n) : U \to \mathbb{T}^n \times B^n,$$

where t is a defining function for Z, such that

$$\omega|_U = \frac{c}{t} d\theta_1 \wedge dt + \sum_{i=2}^{n} d\theta_i \wedge dp_i.$$

Moreover, the functions $t, p_2, \dots, p_n$ depend only on F. The number c is the modular period of the component of Z containing m.

The S^1-valued functions

$$\theta_1, \dots, \theta_r$$

are called angle coordinates *and the $\mathbb{R}$-valued functions*

$$t, p_2, \dots, p_r$$

are called action coordinates*.*

4.1 Outline of the proof

The proof is carried out in several steps:

- we show that the foliation by Liouville tori is trivial on a neighbourhood of a torus, i.e. a product $\mathbb{T}^n \times B^n$ (Section 4.4)
- we prove a "local" action-angle coordinate theorem, the Darboux-Carathéodory theorem (Section 4.5)
- we construct a toric action using the flow of the integrals, whose moment map provides a set of "action" coordinates (Section 4.6)
- and finally we combine the previous steps to obtain the desired action-angle coordinates on a neighbourhood of a Liouville torus (Section 4.7).

4.2 Example: Gluing integrable systems on manifolds with boundary

Let (M, ω) be a b-symplectic manifold with critical hypersurface Z. Restricting the b-symplectic form ω to a component W of $M \backslash Z$, we obtain a symplectic form

$$\omega\big|_W \in \Omega^2(W).$$

The closure of W is a manifold with boundary, and the asymptotics of $\omega\big|_W$ near this boundary can been described in the following way [NT]: For each $p \in \partial W$, there is a neighborhood diffeomorphic to the halfspace $\{(x_1, y_1, \dots, x_n, y_n) \mid x_1 \geq 0\}$ on which

$$\omega = \frac{1}{x_1} dx_1 \wedge dy_1 + \sum_{i>1} dx_i \wedge dy_i.$$

This observation enables us to study manifolds with boundary equipped with a symplectic form in its interior, as long as the symplectic form has the kind of asymptotics described above. By taking the double of such a manifold with boundary, the boundary becomes a hypersurface. The symplectic form then extends to a b-symplectic form on the double.

As an example, consider the upper hemisphere (including the equator) H_+ of S^2. The symplectic form

$$\frac{1}{h} dh \wedge d\theta$$

defined on the interior of H_+ extends to a b-symplectic form on the double of H_+, which is S^2.

This example can be generalized to $H_+ \times M$, where M is any symplectic manifold, endowed with the product symplectic structure. Moreover, any integrable system on the interior of $H_+ \times M$ which has asymptotics compatible with those of ω near the boundary extends to a b-integrable system on the double $S^2 \times M$. For example, if $(f_1, \dots, f_n)$ is an integrable system on M, then the integrable system

$$(\log |h|, f_1, \dots, f_n)$$

on $H_+ \times M$ extends to the b-integrable system $(\log |h|, f_1, \dots, f_n)$ on $S^2 \times M$.

4.3 Counterexample to the existence of action-angle coordinates on a non-orientable manifold

Our definition of b-Poisson manifolds (Definition 2.2.1) contains the assumption that the underlying manifold M is oriented. We could define b-Poisson manifolds in the same way for non-orientable manifolds. However, the action-angle coordinate theorem does not hold in this case.

As a counterexample, consider the Möbius band $\mathbb{R}^2/(x, y) \sim (x + 1, -y)$ with the b-Poisson structure

$$y \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$$

and the function $f = -\log|y|$. Here, the orbits of $X_f = \frac{\partial}{\partial x}$ do not define a trivial fibration semilocally around the orbit $\{y = 0\}$, so there cannot be action-angle coordinates for this example.

Remark. In [KMS] where we first proved the action-angle coordinate theorem for b-symplectic manifolds, we did not assume that the manifold is orientable, but we did assume that the exceptional hypersurface Z has trivial normal bundle, which is enough since it provides us with a global defining function for Z.

4.4 A topological result

Let (M, ω) be a b-symplectic manifold with critical hypersurface Z. We have already seen in Proposition 3.3.3 that any b-integrable system has an equivalent system of the form $(\log|t|, f_2, \dots, f_n)$, where t is a defining function of Z and $f_2, \dots, f_n$ are smooth functions. From now on we will assume that our system is given in this form.

The key result of this section is Proposition 4.4.2, which gives information about the topology of the foliation $\mathcal{F}$ on the set of regular points of the system, whose compact leaves are the Liouville tori (see Section 3.3). It shows that semi-locally around a Liouville torus the foliation is trivial.

We will need the following lemma to describe the foliation $\mathcal{F}$ given by the Hamiltonian vector fields X_{f_i} as the level sets of a (smooth) submersion.

Lemma 4.4.1. *Let $F = (f_1, \dots, f_n)$ be a b-integrable system where $f_1 = \log|t|$ and the other functions are smooth. Then*

$$\tilde{F} := (t, f_2, \dots, f_n) : U_{\mathcal{F}} \to \mathbb{R}^n$$

is a submersion and the level sets of $\tilde{F}$ correspond to foliation $\mathcal{F}$ given by the Hamiltonian vector fields X_{f_i} of the integrals.

Proof. The differentials $d(\log|t|), df_2, \dots, df_n$ are independent as b-form at any regular point $p \in U_{\mathcal{F}}$. We show that this implies that $dt, df_2, \dots, df_n$ are independent as classical de Rham forms at p. For p outside the critical hypersurface Z this is clear. Let $p \in Z \cap U_{\mathcal{F}}$. Recall from Section 2.2.3 that we can identify T_p^*Z with ${}^bT_p^*M$. Therefore, given a linear combination

$$\alpha_1 dt + \sum_{i=2}^{n} \alpha_i df_i = 0 \in T_p^*M$$

with some α_i non-zero, we can apply i^*, where i is the inclusion of Z in M, to obtain a linear combination

$$\sum_{i=2}^{n} \alpha_i df_i = 0 \in T_p^* Z \cong^b T_p^* M.$$

This contradicts the assumption that $d(\log|t|), df_2, \dots, df_n$, and in particular $df_2, \dots, df_n$, are independent at p as b-forms.

We conclude that the components of $\tilde{F}$ are independent and as a consequence the level sets are n-dimensional and define a foliation.

The leaves of the foliation $\mathcal{F}$ of the system have dimension n as well; their tangent spaces are spanned by $X_{f_1}, \dots, X_{f_n}$. Since $dt(X_{f_i}) = 0$ and $df_j(X_{f_i}) = 0$ for $i = 1, \dots, n$ and $j = 2, \dots, n$, we see that they are contained in the tangent spaces of the level sets of $\tilde{F}$. Hence the two foliations are equal. □

Proposition 4.4.2. *Let $m \in Z$ be a regular point of a b-integrable system (M, ω, F), i.e. $m \in \mathcal{U}_{\mathcal{F}}$. Let $\mathcal{F}$ be the foliation induced by the integrable system on $\mathcal{U}_{\mathcal{F}}$. Assume that the integral manifold $\mathcal{F}_m$ through m is compact, i.e. a torus $\mathbb{T}^n$. Then there exists a neighbourhood U of $\mathcal{F}_m$ and a diffeomorphism*

$$\phi : U \simeq \mathbb{T}^n \times B^n,$$

which takes the foliation $\mathcal{F}$ to the trivial foliation $\{\mathbb{T}^n \times \{b\}\}_{b \in B^n}$.

Proof. We follow the proof given in [LMV] (see Proposition 3.2 therein), where the analogous result for general Poisson manifolds is shown.

Let the b-integrable system be given in the form $F = (f_1, \dots, f_n)$ where $f_1 = \log|t|$ is a b-function and the other functions are smooth. By Lemma 4.4.1, the foliation $\mathcal{F}$ of the system can be described by the submersion $\tilde{F} := (t, f_2, \dots, f_n)$.

Then, as in [LMV], we endow M with some Riemannian metric. Using the exponential map we can find for every point $m' \in \mathcal{F}_m$ a neighbourhood $U_{m'}$ of m' in M, a neighbourhood $V_{m'}$ of m' in $\mathcal{F}_m$ and a smooth map $\psi_{m'} : U_{m'} \to \mathcal{F}_{m'} \cap U_{m'}$ such that $\psi_{m'}|_{U_{m'}} = \mathrm{id}_{\mathcal{F}_{m'} \cap U_{m'}}$ and moreover such that for two different points m' and m'' with $U_{m'} \cap U_{m''} \neq 0$ the maps $\psi_{m'}$ and $\psi_{m''}$ coincide on the intersection. Covering $\mathcal{F}_m$ with open neighbourhoods like this and gluing the maps together, we obtain a smooth map

$$\psi : U \to \mathcal{F}_m \cong \mathbb{T}^n$$

on a neighbourhood U of $\mathcal{F}_m$ which restricts to the identity on $\mathcal{F}_m$. We set $\phi := \psi \times \tilde{F}$. Making U smaller if necessary this defines a diffeomorphism

$$\phi = \psi \times \tilde{F} : U \to \mathbb{T}^n \times B^n$$

and we have a commuting diagram

$$
\begin{tikzcd}
U \arrow[r, dashed, "\phi"] \arrow[dr, "\tilde{F}"'] & \mathbb{T}^n \times B^n \arrow[d, "\pi"] \\
& B^n
\end{tikzcd}
\tag{4.1}
$$

where ψ is the canonical projection. In particular, the level sets of $\tilde{F}$ correspond under ϕ to the level sets of ψ, i.e. the tori $\mathbb{T}^n \times \{b\}$, $b \in B^n$. $\square$

Corollary 4.4.3. *Semilocally around a Liouville torus a b-integrable system is isomorphic to*

$$(\mathbb{T}^n \times B^n, \omega, F := (\log|\pi_1|, \pi_2, \ldots, \pi_n), \tag{4.2}$$

where π_i are the projections onto the components of B^n and ω is some b-symplectic structure with critical hypersurface $Z = \{\pi_1 = 0\}$.

Here, "isomorphic" means that there is a b-symplectomorphism which takes the integrals of the original system to F.

Proof. The result follows directly from the diagram in Equation (4.1). $\square$

We conclude that for describing a b-integrable system semilocally around a Liouville torus, we can restrict our attention to the b-integrable system $(\log|\pi_1|, \pi_2, \ldots, \pi_n)$ on the manifold $\mathbb{T}^n \times B^n$ endowed with some b-symplectic form ω whose critical hypersurface is $Z = \{\pi_1 = 0\}$.

At this point we do not yet have more information about ω. The action-angle coordinate theorem will refine the purely topological result obtained here and in particular will put ω in b-Darboux coordinates semilocally around the Liouville torus.

4.5 Darboux-Carathéodory Theorem

Another important result that precedes the proof of the action-angle coordinate theorem is the Darboux-Carathéodory Theorem. It *locally* extends a set of n Poisson commuting and functionally independent b-functions to a "b-Darboux" coordinate system. We first proved this result in [KMS]:

Theorem 4.5.1. *Let (M^{2n}, ω) be a b-symplectic manifold, m be a point on the exceptional hypersurface Z, and $f_1, \ldots, f_n$ be b-functions, defined on a neighbourhood of m, with the following properties*

- $f_1, \dots, f_n$ *Poisson commute*
- $X_{f_1} \wedge \cdots \wedge X_{f_n}$ *is a nonzero section of* $\Lambda^n({}^bTM)$ *at* m.

Then there exist b-functions $(g_1, \dots, g_n)$ *around* m *such that*

$$\omega = \sum_{i=1}^{n} df_i \wedge dg_i.$$

and the vector fields $\{X_{f_i}, X_{g_j}\}_{i,j}$ *commute.*

Moreover, if f_1 *is not a smooth function, i.e.* $f_1 = c \log |t|$ *for some* $c \neq 0$ *and some local defining function* t *of* Z*, then the functions* g_i *can be chosen to be smooth functions for which*

$$(t, f_2, \dots, f_n, g_1, \dots, g_n)$$

are local coordinates around m.

Proof. For this proof, we will adopt the notation that for a 1-form μ, the vector field X_μ is the vector field satisfying $\iota_{X_\mu}\omega = -\mu$. We begin by inductively constructing a collection $\{\mu_1, \dots, \mu_n\}$ of 1-forms with the property that

$$\begin{aligned} \mu_i(X_{f_j}) &= \delta_i^j \\ \mu_i(X_{\mu_j}) &= 0 \qquad \text{for } j < i. \end{aligned}$$

Assume that we have successfully constructed μ_j for $j < i$, and moreover that this construction satisfies

$$0 \neq X_{f_1} \wedge \dots, X_{f_n} \wedge X_{\mu_1} \wedge \cdots \wedge X_{\mu_{i-1}}$$

(as a section of $\Lambda^*({}^bTM)$) locally near m, and consider the problem of constructing μ_i. Let P_i and K_i be the subbundles (of ${}^bTM$ and ${}^bT^*M$ respectively) defined by

$$\begin{aligned} P_i &= \operatorname{span}(\{X_{f_j}, X_{\mu_k}\}_{j \neq i, k < i}) \\ K_i &= \ker(P_i) \end{aligned}$$

The bundle K_i is a codimension-$(n + i - 2)$ subbundle of ${}^bT^*M$. By the inductive hypothesis, X_{f_i} is not contained in P_i, so there is a section μ_i of K_i for which $\mu_i(X_{f_i}) = 1$. The fact that P_i and X_{f_i} are in the kernel of df_i, but X_{μ_i} is not, reveals that $X_{\mu_i} \notin \operatorname{span}(P_i \cup X_{f_i})$, so

$$0 \neq X_{f_1} \wedge \cdots \wedge X_{f_n} \wedge X_{\mu_1} \wedge \cdots \wedge X_{\mu_i},$$

completing the induction.

Because the df_i and μ_i are b-forms, the X_{f_i} and X_{μ_i} are b-vector fields. Using the fact that the f_i functions Poisson commute, and the properties of the μ_i constructed above,

$$\begin{pmatrix} \omega(X_{f_i}, X_{f_j}) & \omega(X_{\mu_i}, X_{f_j}) \\ \omega(X_{f_i}, X_{\mu_j}) & \omega(X_{\mu_i}, X_{\mu_j}) \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

so $\omega = \sum_{i=1}^{n} df_i \wedge \mu_i$. To check that the μ_i are closed (and therefore exact) in a neighbourhood of m, we apply the Cartan formula for the exterior derivative to calculate that

$$\begin{aligned} d\mu_i(X_{f_j}, X_{f_k}) &= X_{f_j}(\mu_i(X_{f_k})) - X_{f_k}\mu_i(X_{f_j}) - \mu_i([X_{f_j}, X_{f_k}]) \\ &= X_{f_j}\delta_i^k - X_{f_k}\delta_i^j - \mu_i(0) = 0 \\ d\mu_i(X_{f_j}, X_{\mu_k}) &= X_{f_j}\mu_i(X_{\mu_k}) - X_{\mu_k}\mu_i(X_{f_j}) - \mu_i([X_{f_j}, X_{\mu_k}]) \\ &= X_{f_j}(0) - X_{\mu_k}\delta_i^j - \mu_i(0) = 0 \\ d\mu_i(X_{\mu_j}, X_{\mu_k}) &= X_{\mu_j}\mu_i(X_{\mu_k}) - X_{\mu_k}\mu_i(X_{\mu_j}) - \mu_i([X_{\mu_j}, X_{\mu_k}]) \\ &= X_{\mu_j}(0) - X_{\mu_k}(0) - \mu_i(0) = 0. \end{aligned}$$

Letting g_i be any local primitive of μ_i yields the first part of the result.

In the case when $f_1 = c \log |t|$ is non-smooth, we can modify our inductive construction of the μ_i so that in addition to requiring that the μ_i be in K_i, we also insist that they be in $T^*M \subseteq {}^bT^*M$. To check that this restriction is consistent with the requirement that $\mu_i(X_{f_i}) = 1$, we must check that the kernel of X_{f_i} is not identically equal to T^*M, i.e. X_{f_i} does not vanish at m when viewed as a section of TM. But this is clear from the fact that X_{f_i} does not vanish at m when viewed as a section of ${}^bTM$, and $0 = \{f_1, f_i\} = \frac{cdt}{t}(X_{f_i})$.

This proves that the g_i can be chosen to be smooth functions. The fact that

$$\{X_t, X_{f_2}, \ldots, X_{f_n}, X_{g_1}, \ldots, X_{g_n}\}$$

pairwise commute follows from the fact that $\{X_{f_i}, X_{g_j}\}_{i,j}$ do, so

$$(t, f_2, \ldots, f_n, g_1, \ldots, g_n)$$

indeed are local coordinates.

$\square$

4.6 The Hamiltonian $\mathbb{T}^n$-action induced by a b-integrable system

By Corollary 4.4.3 we may assume our b-integrable system is of the form given in Equation (4.2),

$$(\mathbb{T}^n \times B^n, \omega, F := (\log|\pi_1|, \pi_2, \dots, \pi_n),$$

where $\pi_1 = 0$ defines the critical hypersurface Z. We want to construct a $\mathbb{T}^n$ action whose orbits are the Liouville tori of F, $\mathbb{T}^n \times \{b\}_{b\in B^n}$. For the sake of simplicity we denote the components of F by f_i as usual and we set $t := \pi_1$.

Clearly, the vector fields $X_{f_1}, \dots, X_{f_n}$ define a $\mathbb{T}^n$ action on each of the Liouville tori individually. However, it is not guaranteed that their flow defines a torus action on all of $\mathbb{T}^n \times B^n$. In this section we construct an equivalent b-integrable system whose fundamental vector fields *do* define a $\mathbb{T}^n$ action on a neighbourhood of $\mathbb{T}^n \times \{0\}$. The argument, which we already presented in [KMS], follows the idea of the analogous result for Poisson manifolds given in [LMV].

4.6.1 Uniformization of periods

We denote by $\Phi^s_{X_{f_i}}$ the time-s-flow of the (b-)Hamiltonian vector fields X_{f_i}. Consider the combined flow of the (b-)Hamiltonian vector fields $X_{f_1}, \dots, X_{f_n}$:

$$\begin{aligned}
&\Phi : \mathbb{R}^n \times (\mathbb{T}^n \times B^n) \to (\mathbb{T}^n \times B^n) \\
&\big((s_1, \dots, s_n), (x, b)\big) \mapsto \Phi^{s_1}_{X_{f_1}} \circ \dots \circ \Phi^{s_n}_{X_{f_n}}((x, b)).
\end{aligned}$$

Because the X_{f_i} are complete and commute with one another, this defines an $\mathbb{R}^n$-action on $\mathbb{T}^n \times B^n$. When restricted to a single orbit $\mathbb{T}^n \times \{b\}$ for some $b \in B^n$, the kernel of this action is a discrete subgroup of $\mathbb{R}^n$, hence a lattice Λ_b. We call Λ_b the **period lattice** of the orbit $\mathbb{T}^n \times \{b\}$. Since the orbit is compact, the rank of Λ_b is n.

The lattice Λ_b will in general depend on b. The process of *uniformization* entails modifying the action so that $\Lambda_b = \mathbb{Z}^n$ for all b. For any $b \in B^{n-1} \times \{0\}$ and any $a_i \in \mathbb{R}$, the vector field $\sum \rho_i X_{f_i}$ on $\mathbb{T}_n \times \{b\}$ is the b-Hamiltonian vector field of the b-function

$$\rho_1 \log|t| + \sum_{i=2}^{n} \rho_i f_i,$$

where we recall that f_i are smooth for $i = 2, \dots, n$. However, by Proposition 4 of [GMPS13], if such a vector field is 1-periodic, then $\rho_1 = \pm c$, where c is the modular period of the component of Z containing m. Therefore, for all $b \in B^{n-1} \times \{0\}$, the lattice Λ_b is contained in $c\mathbb{Z} \times \mathbb{R}^{n-1} \subseteq \mathbb{R}^n$. To perform the uniformization, pick smooth functions

$$(\lambda_1, \lambda_2, \dots, \lambda_n) : B^n \to \mathbb{R}^n$$

such that

- $(\lambda_1(b), \lambda_2(b), \dots, \lambda_n(b))$ is a basis for the period lattice Λ_b for all $b \in B^n$
- λ_i^1 vanishes along $\{0\} \times B^{n-1}$ for $i > 1$, and λ_i^1 equals the modular period c along $\{0\} \times B^{n-1}$. Here, λ_i^j denotes the j^{th} component of λ_i.

Such functions λ_i exist that satisfy the first condition (perhaps after shrinking B^n) by the implicit function theorem, using the fact that the Jacobian of the equation $\Phi(\lambda, m) = m$ is regular with respect to the s variables. The fact that they can be chosen to satisfy the second condition is due to the discussion above.

Using these functions λ_i we define the "uniformized" flow

$$\begin{aligned} &\tilde{\Phi} : \mathbb{R}^n \times (\mathbb{T}^n \times B^n) \to (\mathbb{T}^n \times B^n) \\ &\big((s_1, \dots, s_n), (x, b)\big) \mapsto \Phi\big(\sum_{i=1}^n s_i \lambda_i(c), (x, b)\big). \end{aligned}$$

The period lattice of this $\mathbb{R}^n$ action is constant now, namely $\mathbb{Z}^n$, and hence the action naturally defines a $\mathbb{T}^n$ action.

4.6.2 Moment map of the $\mathbb{T}^n$-action

Let Y_i denote the fundamental vector fields of the $\mathbb{T}^n$ action constructed above. Then

$$Y_i = \sum_{j=1}^n \lambda_i^j X_{f_j}.$$

We want to find b-functions $p_1, \dots, p_n$ such that the corresponding b-Hamiltonian vector fields X_{p_i} are precisely the fundamental vector fields Y_i.

The vector fields Y_i are Poisson. The first step is to show that the Y_i are Poisson vector fields. Our proof is based on the following lemma shown in [LMV] for general Poisson manifolds (see Claim 2 therein):

Lemma 4.6.1. *If $\mathcal{Y}$ is a complete vector field of period one and P is a bivector field for which $\mathcal{L}_{\mathcal{Y}}^2 P = 0$, then $\mathcal{L}_{\mathcal{Y}} P = 0$.*

We use the Cartan formula for b-symplectic forms given in Equation (2.8) to compute $\mathcal{L}_{\mathcal{Y}}^2 \omega = 0$ for $i = 1, \dots, n$:

$$\begin{aligned}
\mathcal{L}_{Y_i}\mathcal{L}_{Y_i}\omega &= \mathcal{L}_{Y_i}(d(\iota_{Y_i}\omega) + \iota_{Y_i} d\omega) \\
&= \mathcal{L}_{Y_i}(d(-\sum_{j=1}^{n} \lambda_i^j df_j)) \\
&= -\mathcal{L}_{Y_i}\left(\sum_{j=1}^{n} d\lambda_i^j \wedge df_j\right) = 0
\end{aligned}$$

In the last equality we used the fact that λ_i^j are constant on the level sets of F. By Lemma 4.6.1 it follows that

$$\mathcal{L}_{Y_i}\omega = 0,$$

so the vector fields Y_i are Poisson vector fields.

The vector fields Y_i are b-Hamiltonian. To show that each $\iota_{Y_i}\omega$ has a ${}^bC^\infty$ primitive, it suffices to show that the smooth part of $[\iota_{Y_i}\omega]$, i.e. the first summand in its image under the Mazzeo-Melrose isomorphism given in Theorem 2.2.17,

$${}^bH^1(\mathbb{T}^n \times B^n) \cong H^1(\mathbb{T}^n \times B^n) \oplus H^0(\mathbb{T}^n \times B^n),$$

is zero. This follows from the fact that the value of the smooth part $H^1(\mathbb{T}^n \times B^n)$ of $\iota_{Y_i}\omega$ is determined by integrating it along loops which are tangent to the fibres. But the kernel of $\iota_{Y_i}\omega$ contains the tangent space to the torus fibres, since the latter are spanned by $X_{f_1}, \dots, X_{f_n}$ and the f_i commute. Therefore these integrals are zero.

We conclude that the fundamental vector fields Y_i are b-Hamiltonian and we denote their Hamiltonian functions by p_i. Because λ_i^1 vanish along $\{0\} \times B^{n-1}$ for $i > 1$, the forms $\iota_{Y_i}\omega = -\sum_{j=1}^n \lambda_i^j df_j$ are smooth for $i > 1$, so the functions p_i are smooth for $i > 1$. Because λ_1^1 equals c along $\{0\} \times B^{n-1}$, p_1 has the form $c \log|t|$ for some defining function t.

Remark. A slight alternation of this proof uses Proposition 2.4.4 about the splitting of Hamiltonian torus actions. Then, in the uniformization step we can choose any functions $\lambda_1, \ldots, \lambda_n$ which pointwise provide a basis of the period lattice. The resulting $\mathbb{T}^n$ action is Hamiltonian by the argument given above, but the moment map, i.e. the set of functions $p_1, \ldots, p_n$ whose Hamiltonian vector fields are the fundamental vector fields of the action, might not "split" in the sense that only a_1 is a b-functions and the others are smooth. However, since the action is Hamiltonian and effective (because of the independence of $X_{f_1}, \ldots, X_{f_n}$), hence toric, we can apply Proposition 2.4.4 to obtain a basis of the Lie algebra of $\mathbb{T}^n$ such that the moment map expressed with respect to this basis splits into a b-function $c \log|t|$ and $n-1$ smooth functions.

4.7 Proof of the action-angle coordinate theorem for b-integrable systems

We will combine the results of the previous sections to prove the first main result of this thesis, the action-angle coordinate theorem, Theorem A. The setting is the same as in the previous section. Let m be a point on a Liouville torus $\mathbb{T}^n \times \{b\}$ lying inside the critical hypersurface (i.e. $b_1 = 0$).

Applying Darboux-Carathéodory. The construction in Section 4.6.2 yields candidates $p_1 = c \log|t|, p_2, \ldots, p_n$ for the "action coordinates". We use the Darboux-Carathéodory theorem to complete these "action coordinates" *locally* around m to a chart

$$(t, p_2, \ldots, p_n, g_1, \ldots, g_n)$$

such that

$$\omega = \frac{c}{t} dg_1 \wedge dt + \sum_{i=2}^{n} dg_i \wedge dp_i.$$

Flowing along the torus. Since the vector fields $Y_i = X_{p_i} = \frac{\partial}{\partial g_i} (i = 1, \ldots, n)$ are the fundamental vector fields of the $\mathbb{T}^n$-action, in the local chart introduced above the flow of the vector fields gives a linear action on the g_i coordinates:

$$(s_1, \ldots, s_n) \cdot (g_1, \ldots, g_n, t, p_2, \ldots, p_n) = (g_1 + s_1, \ldots, g_n + s_n, t, p_2, \ldots, p_n)$$

Therefore, if the functions $g_1, \ldots, g_n$ were initially defined on a neighbourhood U of m, we can extend them to the whole set $U' := \pi^{-1}(\pi(U))$ (i.e. the union of all tori that intersect non-trivially with U). We denote the extensions of these functions by the same symbols.

The vector fields $\frac{\partial}{\partial g_i}$ have period 1 on U, so we can view g_i as S^1 valued coordinates ($i = 1, \ldots, n$). We denote them by the "angle" variable θ_i for this reason.

Action-angle chart. It remains to check that the extended functions

$$(\theta_1, \ldots, \theta_n, t, p_2, \ldots, p_n)$$

define a coordinate system on U' and that ω still has the form

$$\omega = \frac{c}{t} d\theta_1 \wedge dt + \sum_{i=2}^{n} d\theta_i \wedge dp_i. \tag{4.3}$$

It is clear that $\{p_i, \theta_j\} = \delta_{ij}$ on U'. To show that $\{\theta_i, \theta_j\} = 0$, we note that this relation holds on U and flowing with the vector fields X_{p_k} we see that it holds on the whole set U':

$$X_{p_k}\big(\{\theta_i, \theta_j\}\big) = \{\{\theta_i, \theta_j\}, p_k\} = \{\theta_i, \delta_{ij}\} - \{\theta_j, \delta_{ik}\} = 0.$$

This verifies that ω has the form (4.3) above and in particular, we conclude that the derivatives of the functions $t, \theta_1, p_2, \theta_2, \ldots, p_n, \theta_n$ are independent on U, hence these functions define a coordinate system. This completes the proof of the action-angle coordinate theorem.

Chapter 5

Action-angle coordinates for non-commutative b-integrable systems

In the previous chapter we proved the action-angle coordinate theorem for commutative b-integrable systems, Theorem A. Non-commutative integrable systems on general Poisson manifolds were extensively studied in [LMV]. The main result is the action-angle coordinate theorem for these systems which we recalled in Theorem 3.4.3.

In this chapter we will prove the action-angle coordinate theorem for non-commutative b-integrable systems using both the techniques of [LMV] and the theory of b-symplectic manifolds. We will see that a dense subset of the b-symplectic manifold and a dense subset of Z is foliated by r-dimensional tori, where r is the rank of the system.

The difference to the result in the general Poisson case is that we explicitly consider the hypersurface Z where the Poisson structure drops rank and show the existence of action-angle coordinates for Liouville tori contained in Z. The results of this chapter were published in [KMb] (joint work with Eva Miranda).

Theorem B (Action-angle coordinates for non-commutative b-integrable systems)**.** *Let (M,ω) be a b-symplectic manifold with critical hypersurface Z. Let F be a non-commutative b-integrable system on (M,ω) of rank r and let $m \in Z$ be a regular point of the system lying inside the critical hypersurface. Assume that the integral manifold $\mathcal{F}_m$ containing m is compact, i.e. a Liouville torus. Then there exists an open neighbourhood U of the torus $\mathcal{F}_m$ and a diffeomorphism*

$$(\theta_1,\ldots,\theta_r,t,p_2,\ldots,p_r,x_1,\ldots,x_\ell,y_1\ldots,y_\ell) : U \to \mathbb{T}^r \times B^s,$$

where $\ell = n - r = \frac{s-r}{2}$ *and* t *is a defining function of* Z*, such that*

$$\omega|_U = \frac{c}{t} d\theta_1 \wedge dt + \sum_{i=2}^{r} d\theta_i \wedge dp_i + \sum_{k=1}^{\ell} dx_k \wedge dy_k.$$

Moreover, the functions $f_1, \ldots, f_s$ *depend on* $t, p_2, \ldots, p_r, x_1, \ldots, x_\ell, y_1 \ldots, y_\ell$ *only. The number* c *is the modular period of the component of* Z *containing* m*.*

The S^1*-valued functions*

$$\theta_1, \ldots, \theta_r$$

are called angle coordinates*, the* $\mathbb{R}$*-valued functions*

$$t, p_2, \ldots, p_r$$

are called action coordinates *and the remaining* $\mathbb{R}$*-valued functions*

$$x_1, \ldots, x_\ell, y_1 \ldots, y_\ell$$

are called transverse coordinates*.*

The structure of the proof is similar to the commutative case: First, we show a topological result about the foliation induced by the system in the neighbourhood of a Liouville torus (Section 5.2) and a version of the Darboux-Carathéodory theorem (Section 5.4). We perform uniformization of periods to construct a $\mathbb{T}^r$-action whose orbits are the Liouville tori (Section 5.5.1). In contrast to the commutative case, additional arguments involving Cas-basic functions are necessary in the last step where we combine the previous results to prove the existence of action-angle coordinates (Section 5.6).

5.1 Example: Non-commutative integrable systems on manifolds with boundary

In Section 4.2 we constructed b-integrable systems by gluing together integrable systems on manifolds with boundary. We can reproduce a similar procedure in the non-commutative case.

As a concrete example, let the manifold with boundary be $M = N \times H_+$, where (N, ω_N) is any symplectic manifold and H_+ is the upper hemisphere including the equator. We endow the interior of H_+ with the symplectic form $\frac{1}{h} dh \wedge d\theta$, where (h, θ) are the standard height and angle coordinates and the interior of M with the corresponding product structure. Now let $(f_1, \ldots, f_s)$ be a non-commutative integrable system of rank r on N. Then on the interior of M we can, for instance, define the following (smooth) non-commutative integrable system:

$$(\log |h|, f_1, \ldots, f_s)$$

Taking the double of M we obtain a non-commutative b-integrable system on $N \times S^2$.

5.2 A topological result

Let (M, ω) be a b-symplectic manifold with critical hypersurface Z. We have already seen a normal form result in Proposition 3.5.3, according to which any non-commutative b-integrable system of rank r has an equivalent system

$$(f_1 = \log |t|, f_2, \ldots, f_s)$$

where $f_2, \ldots, f_s$ are smooth and $f_1, \ldots, f_r$ commute with all the functions f_i.

As in the commutative case, the following result allows us to describe the foliation $\mathcal{F}$ induced by the Hamiltonian vector fields X_{f_i} as the level sets of a submersion:

Lemma 5.2.1. *Let $F = (f_1, \ldots, f_n)$ be a non-commutative b-integrable system where $f_1 = \log |t|$ and the other functions are smooth. Then*

$$\tilde{F} = (t, f_2, \ldots, f_s) : U_{\mathcal{F}} \to \mathbb{R}^s$$

is a submersion and the level sets of $\tilde{F}$ correspond to foliation $\mathcal{F}$ given by the Hamiltonian vector fields X_{f_i} of the integrals.

As a consequence, we see that around a Liouville torus, the foliation $\mathcal{F}$ is semilocally trivial, i.e. a product $\mathbb{T}^r \times B^s$.

Proposition 5.2.2. *Let $m \in Z$ be a regular point of a non-commutative b-integrable system (M, ω, F), i.e. $m \in \mathcal{U}_F \cap M_{F,r}$. Let $\mathcal{F}$ be the foliation induced by the integrable system on $\mathcal{U}_{\mathcal{F}} \cap M_{F,r}$. Assume that the integral manifold $\mathcal{F}_m$ through m is compact, i.e. a torus $\mathbb{T}^r$. Then there exists a neighbourhood U of $\mathcal{F}_m$ and a diffeomorphism*

$$\phi : U \simeq \mathbb{T}^r \times B^s,$$

which takes the foliation $\mathcal{F}$ to the trivial foliation $\{\mathbb{T}^r \times \{b\}\}_{b \in B^s}$.

Proof. We use Lemma 5.2.1 to view the leaves of the foliation $\mathcal{F}$ as the level sets of the submersion $\tilde{F} = (t, f_2, \ldots, f_s)$.

As described in Proposition 4.4.2, choosing an arbitrary Riemannian metric on M allows us to define a canonical projection $\psi : U \to \mathcal{F}_m \cong \mathbb{T}^r$. Setting $\phi := \psi \times \tilde{F}$ we have a commuting diagram

$$\begin{tikzcd} U \arrow[r, dashed, "\phi"] \arrow[dr, "\tilde{F}"'] & \mathbb{T}^r \times B^s \arrow[d, "\pi"] \\ & B^s \end{tikzcd} \tag{5.1}$$

where

$$\pi = (\pi_1, \ldots, \pi_s) : \mathbb{T}^r \times B^s \to B^s$$

is the canonical projection. □

As a consequence of Proposition 5.2.2, we obtain the following semilocal normal form result for non-commutative b-integrable systems:

Corollary 5.2.3. *Semilocally around a Liouville torus a non-commutative b-integrable system is isomorphic to*

$$(\mathbb{T}^r \times B^s, \omega, G := (\log|\pi_1|, \pi_2, \ldots, \pi_s),$$

where π_i are the projections onto the components of B^s and ω is some b-symplectic structure with critical hypersurface $Z = \{\pi_1 = 0\}$.

Here, "isomorphic" means that there is a b-symplectomorphism which takes the integrals of the original system to a system equivalent to G.

Proof. The result follows from the diagram in Equation (5.1). In particular, the Poisson structure on the target space $V := F(U) \cap \mathbb{R}^s = G(\phi(U)) \cap \mathbb{R}^s$ is preserved: Using the notation of the proof above, and denoting the Poisson bracket associated with the b-symplectic form ω on $\mathbb{T}^r \times B^s$ by $\{\cdot,\cdot\}_{\mathbb{T}^r \times B^s}$, we have

$$\{g, h\}_V = \{g \circ F, h \circ F\} = \{g \circ G \circ \phi, h \circ G \circ \phi\} = \{g \circ G, h \circ G\}_{\mathbb{T}^r \times B^s}.$$

The right hand side is, by definition, the Poisson bracket of g and h induced by G on the target space, which we see coincides with the one induced by F, i.e. $\{\cdot,\cdot\}_V$. □

We conclude that for describing a non-commutative b-integrable system semilocally around a Liouville torus, we can restrict our attention to the non-commutative b-integrable system $(\log|\pi_1|, \pi_2, \dots, \pi_n)$ on the manifold $\mathbb{T}^n \times B^n$ endowed with some b-symplectic form ω whose critical hypersurface is $Z = \{\pi_1 = 0\}$.

At this point we do not yet have more information about ω. The action-angle coordinate theorem will refine the purely topological result obtained here and in particular will put ω in b-Darboux coordinates semilocally around the Liouville torus.

5.3 Casimir-basic functions

We have introduced the notion of Casimir-basic functions for general Poisson manifolds in Section 3.4.2. The notion is based on the Poisson structure induced on the target space of the system. In the b-case, i.e. for non-commutative b-integrable systems, we have defined this structure in a very similar way, see Section 3.5.1. We now introduce Cas-basic functions in this case and prove some basic properties .

Let F be a non-commutative b-integrable system on the b-symplectic manifold (M, ω) with critical hypersurface Z and let $\{\cdot, \cdot\}_V$ be the Poisson bracket induced on $V := F(M) \cap \mathbb{R}^s$.

Definition 5.3.1. An F-basic function on M is a b-function which coincides with a function of the form $g \circ F$ on $M \backslash Z$, where g is a function on V.

We show a characterization of F-basic functions similar to the smooth case using Lemma 5.2.1 of the previous section.

Proposition 5.3.2. *A b-function g is F-basic if and only if $X_{f_i}(g) = 0$ for $i = 1, \dots, r$.*

Proof. We can assume that F is of the form $(f_1 = \log|t|, f_2, \dots, f_s)$ where $f_2, \dots, f_s$ are smooth. As in Lemma 5.2.1, let $\tilde{F} := (t, f_2, \dots, f_s)$. Then g is F-basic if and only if it is $\tilde{F}$-basic i.e. of the form $g = h \circ \tilde{F}$ on M. This is equivalent to g being constant on the level sets of $\tilde{F}$. The latter are spanned by $X_{f_1}, \dots, X_{f_r}$ and hence we can characterize an F-basic functions g by the property that $X_{f_i}(g) = 0$ for $i = 1, \dots, r$. □

The Poisson structure $\{\cdot, \cdot\}_V$ allows us to define the following important class of functions:

Definition 5.3.3. A b-function g on M is said to be a **Casimir-basic function**, or **Cas-basic function** for short, if there exists a Casimir function h on (V, Π_V) such that g coincides with $h \circ F$ on $M \backslash Z$.

The proof of the following proposition, shown in [LMV] for the general Poisson case, works in the same way in the b-case:

Proposition 5.3.4. *A b-function is Cas-basic if and only if it commutes with all F-basic functions.*

Proof. Suppose that $g \in^b C^\infty(M)$ is Cas-basic, $g = h \circ F$ on $M \backslash Z$. Then clearly for an F-basic function $k \circ F$ we have

$$0 = \{h, k\}_V \circ F = \{g, k \circ F\}.$$

To show the other direction, let $g \in^b C^\infty(M)$ be a b-function that is in involution with all F-basic functions. Then $X_{f_i}(g) = \{g, f_i\} = 0$ for $i = 1, \dots, r$, hence g is F-basic, $g = h \circ F$ for some function h on V. Let $k \in C^\infty(V)$; then $k \circ F$ is constant on the fibres of F, so that

$$\{h, k\}_B \circ F = \{g, k \circ F\} = 0,$$

where we have used that F is a Poisson map. It follows that $\{h, k\}_B = 0$ for all functions k on B^s, hence that g $(= h \circ F)$ is Cas-basic. This shows one implication of (1), the other one is clear. $\square$

The following lemma, which was stated and proved in exactly the same way in [LMV], will be used as a tool in the proof of the action-angle coordinate theorem.

Lemma 5.3.5. *Let $F : M \to \overline{R}^s$ be an s-tuple of b-functions on the b-symplectic manifold $M = \mathbb{T}^r \times B^s$. If the coefficients of a vector field of the form $Z = \sum_{j=1}^r \psi_j X_{f_j}$ are F-basic and the vector field has period one, then the coefficients are Cas-basic.*

Proof. We follow the proof in [LMV] replacing the smooth functions f_i by b-functions. Let Z be a vector field as described in the Lemma. We define another vector field

$$Z_0 := \sum_{i=1}^r \psi_i(m) X_{f_i},$$

where m is an arbitrary point in $\mathbb{T}^r \times B_0^s$. Then the restriction of Z_0 to $F^{-1}(F(m))$ is periodic of period 1. Let h be an F-basic function on $\mathbb{T}^r \times B_0^s$, and let us denote the (local) flow of X_h by Φ_t. Since

$$[X_h, Z_0] = \sum_{i=1}^{r} \psi_i(m)[X_h, X_{f_i}] = 0,$$

for $|t|$ sufficiently small, the flow of Z_0 starting from $\Phi_t(m)$ is also periodic of period 1. Since the coefficients of Z are the unique continuous functions such that $Z = Z_0$ on $F^{-1}(F(m))$ and such that the flow of Z from every point has period 1, it follows that $\psi_i(\Phi_t(m)) = \psi_i(m)$ for $|t|$ sufficiently small. Taking the limit $t \to 0$ yields that $X_h(\psi_i) = 0$ for every F-basic function on $\mathbb{T}^r \times B_0^s$. Thus, ψ_i is Cas-basic, for $i = 1, \ldots, r$. $\square$

5.4 Darboux-Carathéodory theorem

Similar to the commutative case (Theorem 4.5.1) we show that we can locally complete a set of k commuting b-functions to a b-Darboux chart. The difference is that now we do not require $k = n$.

Lemma 5.4.1 (Darboux-Carathéodory theorem for b-integrable systems). *Let m be a point lying inside the exceptional hypersurface Z of a b-symplectic manifold (M^{2n}, ω). Let t be a defining function of Z around m and let $f_2, \ldots, f_k$ be C^∞ functions defined on a neighbourhood of m with the following properties*

- *$f_2, \ldots, f_k$ Poisson commute*
- *$X_{f_2} \wedge \cdots \wedge X_{f_k}$ is a nonzero section of $\Lambda^{k-1}({}^bTM)$ at m.*

Then there exist, on a neighborhood U of m, functions

$$g_1, \ldots, g_k, p_1, \ldots, p_{n-k}, q_1, \ldots, q_{n-k},$$

such that

(a) the $2n$ functions $(t, g_1, f_2, g_2, \ldots, f_k, g_k, p_1, q_1, \ldots, p_{n-k}, q_{n-k})$ form a system of coordinates on U centered at m;

(b) the b-symplectic form ω is given on U by

$$\omega = \frac{1}{t} dg_1 \wedge dt + \sum_{i=2}^{k} dg_i \wedge df_i + \sum_{i=2}^{n-k} dp_i \wedge dq_i.$$

Proof. Let us denote the b-Poisson structure dual to ω by Π. From the Darboux-Carathéodory Theorem for non-commutative integrable systems on Poisson manifolds it follows that on a neighborhood U of m we can complete the functions $f_2, \dots, f_k$ to a coordinate system

$$(f_2, g_2, \dots, f_k, g_k, z_1, \dots, z_{2n-2r+2})$$

centred at m such that the b-Poisson structure reads

$$\Pi = \sum_{i=2}^{k} \frac{\partial}{\partial f_i} \wedge \frac{\partial}{\partial g_i} + \sum_{i,j=1}^{2n-2k} \phi_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$$

for some functions ϕ_{ij}. The image of the coordinate functions is an open subset of $\mathbb{R}^{2n}$; we can assume that it is a product $U_1 \times U_2$ where U_2 corresponds to the image of $z_1, \dots, z_{2n-2k}$. Then

$$\Pi_2 = \sum_{i,j=1}^{2n-2r+2} \phi_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$$

is a b-Poisson structure on U_2 and hence by the b-Darboux theorem (Theorem 2.2.10), there exist coordinates on U_2

$$(t, g_1, p_2, q_2, \dots, p_{n-k}, q_{n-k}),$$

where t is the local defining function for Z that we fixed in the beginning, such that

$$\Pi_2 = t \frac{\partial}{\partial g_1} \wedge \frac{\partial}{\partial t} + \sum_{i=2}^{n-r} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i}.$$

The result follows immediately. $\square$

Remark. A more direct proof can be given using the techniques we used in the commutative case, see Theorem 4.5.1.

5.5 The Hamiltonian $\mathbb{T}^r$ action induced by a non-commutative b-integrable system

As in Section 4.6, we use the flow of the b-Hamiltonian vector fields X_{f_i} to construct a Hamiltonian $\mathbb{T}^r$-action on the manifold $\mathbb{T}^r \times B^s$. According to Corollary 5.2.3, for the semilocal study of a non-commutative b-integrable system around a Liouville torus, we can assume that our system is given on $\mathbb{T}^r \times B^s$ with a b-symplectic form whose critical hypersurface is $Z = \{\pi_1 = 0\}$ and that the integrals are $f_1 = \log|\pi_1|, f_2 = \pi_2, \dots, f_s = \pi_s$. Let c be the modular period of Z.

5.5.1 Uniformization of periods

This step is almost identical to the commutative case. For the sake of completeness we restate the main idea of the argument.

Consider the joint flow of the vector fields $X_{f_1}, \ldots, X_{f_r}$ given by

$$\begin{aligned}\Phi &: \mathbb{R}^r \times (\mathbb{T}^r \times B^s) \to \mathbb{T}^r \times B^s \\ \big((s_1, \ldots, s_r), (x, b)\big) &\mapsto \Phi^{s_1}_{X_{f_1}} \circ \cdots \circ \Phi^{s_n}_{X_{f_r}}(x, b).\end{aligned}$$

As discussed in Section 4.6.1, this defines an $\mathbb{R}^r$-action on $\mathbb{T}^r \times B^s$ and the kernel of Φ restricted to an orbit $\mathbb{T}^r \times \{b\}$, $b \in B^s$, is a lattice Λ_b called the *period lattice*. We can find smooth functions (after shrinking the ball B^s if necessary)

$$\lambda_i : B^s \to \mathbb{R}^r, \quad i = 1, \ldots, r$$

such that

- $(\lambda_1(b), \lambda_2(b), \ldots, \lambda_r(b))$ is a basis for the period lattice Λ_b for all $b \in B^s$
- λ_i^1 vanishes along $\{0\} \times B^{s-1}$ for $i > 1$, and λ_1^1 equals the modular period c along $\{0\} \times B^{s-1}$. Here, λ_i^j denotes the j^{th} component of λ_i.

Using these functions λ_i we define the "uniformized" flow

$$\begin{aligned}\tilde{\Phi} &: \mathbb{R}^r \times (\mathbb{T}^r \times B^s) \to (\mathbb{T}^r \times B^s) \\ \big((s_1, \ldots, s_r), (x, b)\big) &\mapsto \Phi\Big(\sum_{i=1}^r s_i \lambda_i(b), (x, b)\Big).\end{aligned}$$

The period lattice of this $\mathbb{R}^r$-action is constant now (namely $\mathbb{Z}^r$) and hence the action naturally defines a $\mathbb{T}^r$ action. In the following we will interpret the functions λ_i as functions on $\mathbb{T}^r \times B^s$ (instead of B^s) which are constant on the tori $\mathbb{T}^r \times \{b\}$.

5.5.2 Moment map of the $\mathbb{T}^r$ action

We denote by $Y_1, \ldots, Y_r$ the fundamental vector fields of the $\mathbb{T}^r$ action given by $\tilde{\Phi}$. Note that $Y_i = \sum_{j=1}^r \lambda_i^j X_{f_j}$. As in Section 4.6.2, the Cartan formula together with Lemma 4.6.1 shows that the Y_i are Poisson vector fields.

We now show that the Y_i are b-Hamiltonian and that their primitives are Cas-basic. Consider the b-one forms

$$\alpha_i := \iota_{Y_i} \omega = -\sum_{j=1}^r \lambda_i^j df_j, \quad i = 1, \ldots, r. \tag{5.2}$$

We know that these forms are closed, since the Y_i are Poisson vector fields. We will show that they are exact, $\alpha_i = -dp_i$, meaning that the Y_i are b-Hamiltonian vector fields with b-Hamiltonian functions $p_i \in^b C^\infty(\mathbb{T}^r \times B^s)$.

A priori we already know that p_i will be a smooth function for $i > 1$, since λ_i^1 vanishes along $\mathbb{T}^r \times \{0\} \times B^{s-1}$ for $i > 1$ and therefore the one-forms α_i defined in Equation (5.2) are smooth for $i > 1$. On the other hand, λ_1^1 equals the modular period c along $\mathbb{T}^r \times \{0\} \times B^{s-1}$ and therefore $p_1 = c \log |t|$ for some defining function t.

Homotopy formula. We compute the functions $p_2, \dots, p_r$ explicitly by applying a homotopy formula to the smooth one-forms $\alpha_2, \dots, \alpha_r$. This not only shows that these one-forms are exact but moreover we will conclude that $p_2, \dots, p_r$ are Cas-basic. (For the b-function $p_1 = c \log |t|$ this is clear.)

We state the homotopy formula in its general form as a lemma, the proof can be found in [GS77] (p. 110):

Lemma 5.5.1. [GS77] *Let $Y \subset M$ be an embedded submanifold and suppose that ϕ_t is a smooth retraction of M onto Y. Then for any one-form $\alpha \in \Omega(M)$ we have*

$$\alpha - \phi_0^*(\alpha) = I(d(\alpha_i)) + d(I(\alpha_i)), \quad i = 2, \dots, r,$$

where I is the functional

$$I(\alpha_i) = \int_0^1 \phi_t^*(\iota_{\xi_t}(\alpha))$$

and ξ_t is the tangent vector field along ϕ_t.

We now apply the above homotopy formula to the smooth one-forms $\alpha_2, \dots, \alpha_n$ and the retraction ϕ_τ of $\mathbb{T}^r \times B^s$ to $\mathbb{T}^r \times \{0\} \times B^{s-r}$:

$$\phi_\tau(x_1, \dots, x_r, b_1, \dots, b_r, b_{r+1}, \dots, b_s) = (x, \tau b_1, \dots \tau b_r, b_{r+1}, \dots, b_s).$$

Note that $\phi_0^*(\alpha_i)$ is zero, i.e. for any vector field $X \in \mathcal{X}(\mathbb{T}^r \times \{0\} \times B^{s-r})$, $\alpha_i(X) = 0$. This is because α_i is a linear combination of $d\pi_2, \dots, d\pi_r$ and therefore evaluates to zero for X a linear combination of

$$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_r}, \frac{\partial}{\partial \pi_{r+1}}, \dots, \frac{\partial}{\partial \pi_s}.$$

Inserting into the homotopy formula we have

$$\alpha_i - \underbrace{\phi_0^*(\alpha_i)}_{=0} = I(\underbrace{d(\alpha_i)}_{=0}) + d(I(\alpha_i)), \quad i = 2, \dots, r.$$

Therefore the homotopy formula tells us that the one-forms α_i are exact and that, moreover, the primitive of α_i $(i = 2, \dots, r)$ is explicitly given by $I(\alpha_i)$:

$$I(\alpha_i) = \int_0^1 \phi_\tau^*(\iota_{\xi_\tau}(\alpha_i)).$$

We set $p_i := -I(\alpha_i)$, the Hamiltonian function of the fundamental vector field Y_i associated with α_i. We want to show that these functions p_i are Cas-basic and therefore compute the expression in the integral. The vector field ξ_τ is:

$$\xi_\tau = \frac{d\phi_\tau}{d\tau} \circ \phi_\tau^{-1} = \frac{1}{\tau} \sum_{k=1}^{s} \pi_k \frac{\partial}{\partial \pi_k}.$$

Therefore we have

$$\iota_{\xi_\tau}(\alpha_i) = \frac{1}{\tau} \sum_{j=2}^{r} \lambda_i^j d\pi_j(\xi_\tau) = \frac{1}{\tau} \sum_{j=2}^{r} \sum_{k=1}^{s} \lambda_i^j \pi_k d\pi_j \left(\frac{\partial}{\partial \pi_k} \right) = \frac{1}{\tau} \sum_{j=2}^{r} \lambda_i^j \pi_j.$$

In the last equality we have used $d\pi_j(\frac{\partial}{\partial \pi_k}) = \delta_{jk}$ for $j > 2$. The projections $\pi_j, j = 1, \dots, r$, are obviously Cas-basic. The functions λ_i^j are Cas-basic by Lemma 5.3.5. The pullback ϕ_τ^* does not affect the Cas-basic property since it leaves the non-commutative part of the system invariant. Therefore the functions $\phi_\tau^*(\iota_{\xi_\tau}(\alpha_i))$ and hence $p_2, \dots, p_r$ are Cas-basic.

5.6 Proof of the action-angle coordinate theorem for non-commutative b-integrable systems

We continue in the setting of the previous section. Let $m \in \mathbb{T}^r \times \{0\}$. We want to show the existence of action-angle coordinates semilocally around the Liouville torus passing through m.

Applying Darboux-Carathéodory. As a first, local, step we apply the Darboux-Carathéodory theorem for non-commutative b-integrable systems to the independent commuting smooth functions $p_2, \dots, p_n$. Then on a neighbourhood U of m we obtain a set of coordinates

$$(t, g_1, p_2, g_2, \dots, p_r, g_r, x_1, y_1, x_2, y_2, \dots, x_\ell, y_\ell),$$

where $\ell = (s-2r)/2$, such that

$$\omega|_U = \frac{c}{t} dg_1 \wedge dt + \sum_{i=2}^{k} dp_i \wedge dg_i + \sum_{i=1}^{\ell} dx_i \wedge dy_i. \tag{5.3}$$

Flowing along the torus. The idea of the next steps is to extend this local expression to a neighbourhood of the Liouville torus using the $\mathbb{T}^r$-action given by the vector fields X_{p_k}. First, note that the functions $(x_1, y_1, x_2, y_2, \dots, x_\ell, y_\ell)$ do not depend on f_i and therefore can be extended to the saturated neighborhood $W := \pi^{-1}(\pi(U))$. Note that $Y_i = \frac{\partial}{\partial g_i}$ and therefore the flow of the fundamental vector fields of the Y_i-action corresponds to translations in the g_i-coordinates. In particular, we can naturally extend the functions g_i to the whole set W as well.

Action-angle chart. We show that the functions

$$t, g_1, p_2, g_2, \dots, p_r, g_r, x_1, y_1, x_2, y_2, \dots, x_\ell, y_\ell \tag{5.4}$$

which are defined on W, indeed define a chart there (i.e. they are independent) and that ω still has the form given in Equation (5.3).

It is clear that $\{p_i, g_j\} = \delta_{ij}$ on W. To show that $\{g_i, g_j\} = 0$, we note that this relation holds on U and flowing with the vector fields X_{p_k} we see that it holds on the whole set W:

$$X_{p_k}\big(\{g_i, g_j\}\big) = \{\{g_i, g_j\}, p_k\} = \{g_i, \delta_{ij}\} - \{g_j, \delta_{ik}\} = 0.$$

This verifies that ω has the form (5.3) above and in particular, we conclude that the derivatives of the functions (5.4) are independent on W, hence these functions define a coordinate system.

Since the vector fields $\frac{\partial}{\partial g_i}$ have period one, we can view $g_1, \dots, g_r$ as $\mathbb{R}\backslash\mathbb{Z}$-valued functions ("angles") and therefore use the letter θ_i instead of g_i.

This finishes the proof of the action-angle coordinate theorem in the case of non-commutative b-integrable systems.

Chapter 6

Cotangent models and examples of integrable systems

The classical action-angle coordinate theorem identifies a neighbourhood of a Liouville torus of an integrable system with the manifold $\mathbb{T}^n \times \mathbb{R}^n$ endowed with the standard symplectic structure and such that the integrals are functions of the canonical projections onto the components of $\mathbb{R}^n$.

On the other hand, consider the action of $\mathbb{T}^n$ on $\mathbb{T}^n \times \mathbb{R}^n$ by translations on the $\mathbb{T}^n$ component. Then the canonical projections onto the $\mathbb{R}^n$ component in the product are precisely the components of the *moment map* of this action with respect to the canonical basis of the Lie algebra of $\mathbb{T}^n$. Moreover, identifying $\mathbb{T}^n \times \mathbb{R}^n$ with $T^*\mathbb{T}^n$, we can understand the action of $\mathbb{T}^n$ on $\mathbb{T}^n \times \mathbb{R}^n$ as the *cotangent lift* of the action of $\mathbb{T}^n$ on itself by translations. In summary, an integrable system on a symplectic manifoldh can be viewed semilocally as the moment map of the cotangent lift of the action of $\mathbb{T}^n$ on itself.

In this chapter we will formalize this idea and extend it to (non-commutative) b-integrable systems. We will first introduce the required concepts, such as cotangent lifts, and finish the chapter with several examples of (non-commutative) b-integrable systems that can be constructed using the viewpoint of torus actions. The results of this chapter were published in [KMa] (joint work with Eva Miranda).

6.1 Cotangent lifts and b-cotangent lifts

In this section we work towards the definition of *cotangent models* for integrable systems on symplectic and b-symplectic manifolds. The standard definition of cotangent lifts on symplectic manifolds was reviewed in the Preliminaries (Section 2.5).

6.1.1 Symplectic cotangent lift of translations on the torus

Consider the action of the torus $\mathbb{T}^n$ on itself by translations,

$$\tau_\beta : \mathbb{T}^n \to \mathbb{T}^n : \theta \mapsto \theta + \beta, \quad \beta \in \mathbb{T}^n.$$

We want to explicitly compute the moment map for the Hamiltonian action obtained by lifting this action to the cotangent bundle $T^*\mathbb{T}^n$. Let $\theta_1, \ldots, \theta_n$ be the standard (S^1-valued) coordinates on $\mathbb{T}^n$ and let

$$\underbrace{\theta_1, \ldots, \theta_n}_{=:\theta}, \underbrace{p_1, \ldots, p_n}_{:=p} \tag{6.1}$$

be the corresponding chart on $T^*\mathbb{T}^n$, i.e. we associate to the coordinates (6.1) the cotangent vector $\sum_i p_i d\theta_i \in T^*_\theta\mathbb{T}^n$. The Liouville one-form, which we defined intrinsically in Equation (2.12), is given in these coordinates by

$$\lambda = \sum_{i=1}^n p_i d\theta_i$$

and its negative differential is the standard symplectic form on $T^*\mathbb{T}^n$:

$$\omega_{can} = \sum_{i=1}^n d\theta_i \wedge dp_i. \tag{6.2}$$

The lift of the translation τ_β to $T^*\mathbb{T}^n$ is given by

$$\hat{\tau}_\beta : (\theta, p) \mapsto (\theta + \beta, p).$$

The moment map $\mu_{can} : T^*\mathbb{T}^n \to \mathfrak{t}^*$ of the lifted action with respect to the canonical symplectic form is

$$\mu_{can}(\theta, p) = \sum_i p_i d\theta_i, \tag{6.3}$$

where the θ_i on the right hand side are understood as elements of $\mathfrak{t}^*$ in the obvious way.

Even simpler, if we identify $\mathfrak{t}^*$ with $\mathbb{R}^n$ by choosing the standard basis $\frac{\partial}{\partial\theta_1}, \ldots, \frac{\partial}{\partial\theta_n}$ of $\mathfrak{t}$ then the moment map is just the projection onto the second component of $T^*\mathbb{T}^n \cong \mathbb{T}^n \times \mathbb{R}^n$. We will adopt this viewpoint from now on and simply write

$$\mu = (p_1, \ldots, p_n).$$

Note that the components of μ naturally define an integrable system on $T^*\mathbb{T}^n$.

6.1.2 b-Cotangent lifts of $\mathbb{T}^n$

As before, let $T^*\mathbb{T}^n$ be endowed with the standard coordinates (θ, p), $\theta \in \mathbb{T}^n$, $p \in \mathbb{R}^n$ and consider again the $\mathbb{T}^n$-action on $T^*\mathbb{T}^n$ induced by lifting translations of the torus $\mathbb{T}^n$. We now want to view this action as a b-Hamiltonian action with respect to a suitable b-symplectic form.

In analogy to the classical Liouville one-form we define the following non-smooth one-form away from the hypersurface $Z = \{p_1 = 0\}$:

$$c \log |p_1| d\theta_1 + \sum_{i=2}^{n} p_i d\theta_i.$$

The negative differential of this form extends to a b-symplectic form on $T^*\mathbb{T}^n$, which we call the **twisted b-symplectic form** on $T^*\mathbb{T}^n$ (we will explain the terminology below):

$$\omega_{tw,c} := \frac{c}{p_1} d\theta_1 \wedge dp_1 + \sum_{i=2}^{n} d\theta_i \wedge dp_i. \tag{6.4}$$

Here, c is the modular period of Z. The moment map of the lifted action with respect to this b-symplectic form is then given by

$$\mu_{tw,c} := (c \log |p_1|, p_2, \ldots, p_n), \tag{6.5}$$

where we identify $\mathfrak{t}^*$ with $\mathbb{R}^n$ as before.

We call this lift together with the b-symplectic form (6.4) the **twisted b-cotangent lift** with modular period c. Note that, in analogy to the symplectic case, the components of the moment map define a b-integrable system on $(T^*\mathbb{T}^n, \omega_{tw,c})$.

Remark. We use the term "twisted b-symplectic form" to distinguish our construction from the canonical b-symplectic form on ${}^bT^*M$, where M is any smooth manifold. The latter is obtained naturally if we imitate the symplectic approach and interpret the intrinsic definition of the Liouville one-form (2.12) in the b-setting (see e.g. [NT]). This means that for (M, Z) a b-manifold, we define a b-form λ on ${}^bT^*M$ via

$$\langle \lambda_m, v \rangle := \langle m, (\pi_m)_*(v) \rangle, \tag{6.6}$$

where $v \in^b T({}^bT^*M)$, $m \in^b T^*M$ and

$$\pi : {}^bT^*M \to M$$

is the canonical projection. Then the negative differential

$$\omega = -d\lambda$$

is the "canonical" b-symplectic form on ${}^bT^*M$ where we view ${}^bT^*M$ as a b-manifold with hypersurface $\pi^{-1}(Z)$. To see this, consider a local set of coordinates $x_1, \dots, x_n$ on M, where x_1 is a defining function for Z and consider the corresponding chart

$$(x_1, \dots, x_n, p_1, \dots, p_n)$$

on T^*M, given by identifying the $2n$-tuple above with the b-cotangent vector

$$p_1 \frac{dx_1}{x_1} + \sum_{i=2}^{n} p_i dx_i \in^b T_x^* M.$$

In these coordinates

$$\lambda = p_1 \frac{dx_1}{x_1} + \sum_{i=2}^{n} p_i dx_i \in^b T^*({}^bT^*M).$$

Note that in the present case of the "canonical" b-cotangent lift, the singularity is given by the coordinate x_1 on the base manifold whereas in our "twisted" construction it is given by a fiber coordinate, which is what we require for the description of b-integrable systems. Therefore, only the twisted b-cotangent lift will have applications in the rest of this thesis.

6.1.3 b-Cotangent lifts in the general setting

Above we focused on the case where the manifold M is a torus and the action consists of rotations of the torus on itself, since this is the model that describes (b-)integrable systems semilocally around a Liouville torus.

To obtain a wider class of examples, we now consider any manifold M and the action of any Lie group G on M:

$$\rho : G \times M \to M : (g, m) \mapsto \rho_g(m).$$

As described in Section 6.1 we can lift ρ to an action $\hat{\rho}$ on T^*M, which is Hamiltonian with respect to the standard symplectic structure on T^*M. We want to investigate modifications of this construction. This will allow us to take the reverse route and construct examples of (b-)integrable systems using cotangent lifts of actions as a starting point (Section 6.3).

Canonical b-cotangent lift

Connecting with Remark 6.1.2, we first want to use the canonical b-symplectic structure on ${}^bT^*M$ and study the properties of the b-cotangent lift obtained in this setting.

Let M be an n-dimensional b-manifold with critical hypersurface Z. Consider the b-cotangent bundle ${}^bT^*M$ endowed with the canonical b-symplectic structure as described in Remark 6.1.2. Moreover, assume that the action of G on M preserves the hypersurface Z, i.e. ρ_g is a b-map for all $g \in G$. Then the lift of ρ to an action on ${}^bT^*M$ is well-defined:

$$\hat{\rho} : G \times {}^bT^*M \to {}^bT^*M : (g, m) \mapsto \rho^*_{g^{-1}}(m).$$

We call this action on ${}^bT^*M$, endowed with the canonical b-symplectic structure, the **canonical b-cotangent lift**.

Proposition 6.1.1. *The canonical b-cotangent lift is Hamiltonian with equivariant moment map given by*

$$\mu : {}^bT^*M \to \mathfrak{g}^*, \quad \langle \mu(m), X \rangle := \langle \lambda_m, X^\#|_m \rangle = \langle m, X^\#|_{\pi(m)} \rangle, \tag{6.7}$$

where $m \in {}^bT^*M$, $X \in \mathfrak{g}$, $X^\#$ is the fundamental vector field of X under the action on ${}^bT^*M$ and the function $\langle \lambda, X^\# \rangle$ is smooth because $X^\#$ is a b-vector field.

Proof. The proof of Equation (6.7) for the moment map is exactly the same as in the symplectic case: Using the implicit definition of λ, Equation (6.6), we show that λ is invariant under the action:

$$\langle (\hat{\rho}^*_g \lambda)_m, v \rangle = \langle \lambda_{\hat{\rho}_g(m)}, (\hat{\rho}_g)_* v \rangle = \langle \hat{\rho}_g(m), (\pi_{\hat{\rho}_g(m)})_* ((\hat{\rho}_g)_* v) \rangle =$$

$$= \langle \rho^*_{g^{-1}}(m), (\rho_{g^{-1}})_* ((\pi_m)_*(v)) \rangle = \langle m, (\pi_m)_*(v) \rangle.$$

In going from the first to the second line we have used the definition of $\hat{\rho}$ and applied the chain rule to $\pi_{\hat{\rho}_g(pm)} \circ \hat{\rho}_g = \rho_{g^{-1}} \circ \pi_m$.

Hence we have $\mathcal{L}_{X^\#} \lambda = 0$ and applying the Cartan formula for b-symplectic forms, Equation (2.8), we obtain

$$\iota_{X^\#} \omega_m = -\iota_{X^\#} d\lambda_m = d(\iota_{X^\#} \lambda_m),$$

which proves the expression for the moment map stated above.

Equivalriance of μ is a consequence of the invariance of λ:

$$\langle (Ad^*_{g^{-1}} \circ \mu)(m), X\rangle = \langle \mu(m), Ad_{g^{-1}}X\rangle = \langle \lambda_m, \underbrace{(Ad_{g^{-1}}X)^{\#}}_{=(\hat{\rho}_g)_* X^{\#}}|_m\rangle =$$
$$= \langle \hat{\rho}_g^* \lambda_m, X^{\#}|_{\hat{\rho}_{g^{-1}}(m)}\rangle = \langle \lambda_{\hat{\rho}_{g^{-1}}(m)}, X^{\#}|_{\hat{\rho}_{g^{-1}}(m)}\rangle = \langle \mu(\hat{\rho}_{g^{-1}}(m)), X\rangle$$

for all $g \in G$, $X \in \mathfrak{g}$, $m \in T^*M$, where in the first equality of the second line we have used that λ is invariant. $\square$

Remark. The condition that the action preserves Z means that all fundamental vector fields are tangent to Z and therefore at a point in Z the maximum number of independent fundamental vector fields is $n-1$. This means that the moment map of such an action never defines a b-integrable system on ${}^bT^*M$ since this would require n independent functions.

Twisted b-cotangent lift

We have already defined the twisted b-cotangent lift on the cotangent space of a torus $T^*\mathbb{T}^n$ in Section 6.1.2. In particular, on T^*S^1 with standard coordinates (θ, p) we have the logarithmic Liouville one-form $\lambda_{tw,c} = \log|p| d\theta$ for $p \neq 0$.

Now consider any $(n-1)$-dimensional manifold N and let λ_N be the standard Liouville one-form on T^*N. We endow the product $T^*(S^1 \times N) \cong T^*S^1 \times T^*N$ with the product structure $\lambda := (\lambda_{tw,c}, \lambda_N)$ (defined for $p \neq 0$). Its negative differential $\omega = -d\lambda$ is a b-symplectic structure with critical hypersurface given by $p = 0$.

Let K be a Lie group acting on N and consider the component-wise action of $G := S^1 \times K$ on $M := S^1 \times N$ where S^1 acts on itself by rotations. We lift this action to T^*M as described in the beginning of this section. This construction, where T^*M is endowed with the b-symplectic form ω, is called the **twisted b-contangent lift**.

If $(x_1, \ldots, x_{n-1})$ is a chart on N and $(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1})$ the corresponding chart on T^*N we have the following local expression for λ

$$\lambda = \log|p| d\theta + \sum_{i=1}^{n-1} y_i dx_i.$$

Just as in the symplectic case and in the case of the canonical b-cotangent lift, this action is Hamiltonian with moment map given by contracting the fundamental vector fields with λ:

Proposition 6.1.2. *The twisted b-cotangent lift on $M = S^1 \times N$ is Hamiltonian with equivariant moment map μ given by*

$$\langle \mu(m), X \rangle := \langle \lambda_m, X^{\#}|_m \rangle,$$

where $X^{\#}$ is the fundamental vector field of X under the action on T^*M.

Proof. As in the proof of Proposition 6.1.1, we show that the action preserves the logarithmic Liouville one-form $\lambda = (\lambda_{tw,c}, \lambda_N)$. Since the action splits this amounts to showing invariance of $\lambda_{tw,c}$ under S^1; the invariance of λ_N under K is the classical symplectic result. The former is easy to see:

$$(\hat{\tau})^*_{\varphi}\lambda_{tw,c} = \log|a \circ \hat{\tau}_{\varphi}| d(\underbrace{\theta \circ \hat{\tau}_{\varphi}}_{=\theta+\varphi}) = \log|a| d\theta,$$

where τ is the action of S^1 on itself by rotations and $\varphi \in S^1$. This shows that $\mathcal{L}_{X^{\#}}\lambda = 0$ and as before we conclude the proof by using Cartan's formula. □

A consequence of the equivariance of the moment map is the following:

Corollary 6.1.3. *The moment map of the twisted b-cotangent lift is a Poisson map.*

Remark. A special case of a manifold $S^1 \times N$ is a cylinder $\mathbb{T}^k \times \mathbb{R}^{n-k}$. We will use this construction in Section 6.3.2.

In order to compute the moment map it is convenient to observe that the expression $\langle \lambda, X^{\#} \rangle$ remains unchanged when we replace the fundamental vector field $X^{\#}$ of the action on T^*M by any vector field on T^*M that projects to the same vector field on M (namely the fundamental vector field of the action on M). This follows immediately from the definition of λ, since the expression only contains the differentials of the coordinates of the base manifold M.

6.2 Cotangent models for integrable systems

In this section we give cotangent models for integrable systems on symplectic and b-symplectic manifolds. done for Hamiltonian actions in [Marl] and [GS84]. We keep the convention of denoting a (b-)integrable system by a triple (M, ω, F) where M is a manifold, ω a (b-)symplectic form and F the set of integrals.

With the notation introduced in the the previous sections about cotangent lifts and their moment maps (Sections 6.1.1 and 6.1.2), we define the following models of integrable systems:

1. $$(T^*\mathbb{T}^n)_{can} := (T^*\mathbb{T}^n, \omega_{can}, \mu_{can}) \tag{6.8}$$
   where ω_{can} and μ_{can} are defined in Equations (6.2) and (6.3) respectively.

2. $$(T^*\mathbb{T}^n)_{tw,c} := (T^*\mathbb{T}^n, \omega_{tw,c}, \mu_{tw,c}) \tag{6.9}$$
   where ($\omega_{tw,c}$ and $\mu_{tw,c}$) are defined in Equations (6.4) and (6.5) respectively.

In the next sections we will see that these models can be employed to semilocally describe integrable resp. b-integrable systems around a Liouville torus.

6.2.1 Symplectic case

We formulate the Liouville-Mineur-Arnold theorem (Theorem 3.1.2) in terms of the symplectic cotangent model:

Theorem 6.2.1. *Let $F = (f_1, \dots, f_n)$ be an integrable system on the symplectic manifold (M, ω). Then semilocally around a regular Liouville torus the system is equivalent to the cotangent model $(T^*\mathbb{T}^n)_{can}$ restricted to a neighbourhood of the zero section $(T^*\mathbb{T}^n)_0$ of $T^*\mathbb{T}^n$.*

Proof. Let $\mathcal{T}$ be a regular Liouville torus of the system. The action-angle coordinate theorem (Theorem 3.1.2) implies that there exists a neighbourhood U of $\mathcal{T}$ and a symplectomorphism

$$\psi : U \to (\mathbb{T}^n \times B^n, \omega_{can})$$

such that the "action coordinates", i.e. the projections onto B^n, depend only on the integrals $f_1, \dots, f_n$, hence their composition with ψ yields an equivalent integrable system on U. We know that the projections onto B^n correspond to the moment map μ_{can} of the cotangent lifted action on $T^*\mathbb{T}^n \cong \mathbb{T}^n \times \mathbb{R}^n$ (restricted to $\mathbb{T}^n \times B^n$ and understood with respect to the canonical basis on $\mathfrak{t}^*$), hence we can write

$$\begin{tikzcd} U \arrow[r, "\psi"] \arrow[dr, "F"'] & (T^*\mathbb{T}^n, \omega_{can}) \arrow[d, "\varphi\circ\mu"] \\ & \mathbb{R}^n \end{tikzcd}$$

where φ is the map that establishes the dependence of the action coordinates on $f_1, \dots, f_n$. $\square$

6.2.2 b-symplectic case

The model of the twisted b-cotangent lift, Equation (6.9), allows us to express the action-angle coordinate theorem for b-integrable systems in the following way:

Theorem C. *Let $F = (f_1, \dots, f_n)$ be a b-integrable system on the b-symplectic manifold (M, ω). Then semilocally around a regular Liouville torus $\mathcal{T}$, which lies inside the exceptional hypersurface Z of M, the system is equivalent to the twisted b-cotangent lift model $(T^*\mathbb{T}^n)_{tw,c}$ restricted to a neighbourhood of $(T^*\mathbb{T}^n)_0$. Here c is the modular period of the connected component of Z containing $\mathcal{T}$.*

Proof. The proof is the same as above using the action-angle coordinate theorem for b-integrable systems (Theorem A): Around the Liouville torus $\mathcal{T}$ we have a Poisson diffeomorphism

$$\psi : U \to \mathbb{T}^n \times B^n$$

taking the b-symplectic form on U to

$$\sum_{i=1}^{n-1} d\theta_i \wedge dp_i + \frac{c}{p_n} d\theta_n \wedge dp_n,$$

where $(\theta_1, \dots, \theta_n, p_1, \dots, p_n)$ are the standard coordinates on $\mathbb{T}^n \times B^n$, and such that $p_1, \dots, p_n$ only depend on the integrals. Hence in the language of Section 6.1.2 we have a commuting diagram

$$\begin{tikzcd} U \arrow[r, "\psi"] \arrow[dr, "F"'] & (T^*\mathbb{T}^n, \omega_{tw,c}) \arrow[d, "\varphi\circ\mu_{tw,c}"] \\ & \mathbb{R}^n \end{tikzcd}$$

$\square$

Remark. Observe that the model contains the modular period c of the Poisson manifold. This constant is indeed an intrinsic invariant of the Poisson structure [GMPS13].

6.3 Constructing examples on b-symplectic manifolds

We can use the viewpoint of cotangent lifts not only to describe (b-)integrable systems but also to construct new ones via the following theorem:

Theorem 6.3.1. *Let M be a smooth manifold of dimension n and let G be an n-dimensional* abelian *Lie group acting on M effectively. Pick a basis $X_1, \ldots, X_n$ of the Lie algebra of G. Consider the moment map $\mu : T^*M \to \mathfrak{g}^*$ of one of the following Hamiltonian actions:*

1. *the symplectic cotangent lift on T^*M.*
2. *the twisted b-cotangent lift on T^*M as described in Section 6.1.3, i.e. we assume that M has the form $M = S^1 \times N$ where N is an $(n-1)$-dimensional manifold and that $G = S^1 \times K$ where K an $(n-1)$-dimensional Lie group and that the action splits into an action of S^1 on itself by rotations and an action of K on N.*

Then the components of the moment map with respect to the basis X_i define 1. an integrable resp. 2. a b-integrable system on T^*M.

Proof. Denote the components of the moment map by $f_i := \langle \mu, X_i \rangle$. Effectiveness of the action implies that the f_i are linearly independent everywhere. Moreover, μ is a Poisson map as a consequence of being equivariant (see Section 2.5 and Corollary 6.1.3). Now since the elements X_i commute and μ preserves the Poisson structure, the components f_i of the moment map commute, $\{f_i, f_j\} = 0$. Hence we have obtained a set of n commuting independent functions. □

6.3.1 The geodesic flow

A special case of an S^1-action is obtained in the case of so-called **P-manifolds**. These are Riemannian manifolds which have the property that all their geodesics are closed. For a P-manifold M it can be shown that the geodesics admit a common period (see e.g. [Be], Lemma 7.11); hence their flow induces an S^1-action on M and we can use the twisted b-cotangent lift to obtain a b-Hamiltonian S^1-action on T^*M. In dimension two, examples of P-manifolds are Zoll and Tannery surfaces (see Chapter 4 in [Be]). Given an S^1-action on such a surface, via the cotangent lift we immediately obtain examples of (b-)integrable systems on its cotangent bundle.

6.3.2 Affine manifolds

A smooth manifold M is called *flat* if it admits a flat (i.e. zero curvature) connection. It is called *affine* if moreover the connection is torsion-free.

It is well-known that a simply connected flat manifold is parallelizable, i.e. it admits a basis of vector fields that are everywhere independent. Such a basis is called parallel. The relation between flatness (in the sense that the curvature is zero) and parallelizability was studied in [T]. We are not assuming that the affine manifold is compact.

Bieberbach [Bi] proved in 1911 that any complete affine Riemannian manifold is a finite quotient of $\mathbb{R}^k \times \mathbb{T}^{n-k}$.

Theorem 6.3.2. *Let M be a cylinder $\mathbb{R}^k \times \mathbb{T}^{n-k}$. Then for any choice of parallel basis $X_1, \dots X_n$, we obtain a (b-)integrable system on T^*M.*

Proof. Let $X_1, \dots, X_n$ be a global basis of parallel vector fields. Since the torsion of the connection is zero and the vector fields X_i are parallel, the expression $\nabla_{X_i} X_j - \nabla_{X_j} X_i - [X_i, X_j] = T^\nabla(X_i, X_j) = 0$ yields $[X_i, X_j] = 0$. In other words, the flows of the vector fields commute. Let us denote by $\Phi^{s_j}_{X_j}$ the s_j-time flow of the vector field X_j. Since the manifold is complete, the joint flow of the vector fields X_i then defines an $\mathbb{R}^n$-action[1],

$$\begin{aligned} \Phi : \mathbb{R}^n \times M &\to M \\ \big((s_1, \dots, s_n), (x)\big) &\mapsto \Phi^{s_1}_{X_1} \circ \dots \circ \Phi^{s_n}_{X_n}((x)). \end{aligned}$$

By the construction defined in Section 6.1.3 we obtain a (b-)Hamiltonian action on T^*M and the components of the moment map of this action define a (b-)integrable system (Theorem 6.3.1). □

Remark. We proved the above result only for cylinders $\mathbb{R}^k \times \mathbb{T}^{n-k}$. It would be interesting to explore whether a similar construction is possible for finite quotients of $\mathbb{R}^k \times \mathbb{T}^{n-k}$, which by Bieberbach's result would correspond to all complete affine Riemannian manifolds.

[1] Depending on the topology of the fiber, this action may descend to a $\mathbb{T}^n$-action or more generally to a $\mathbb{R}^k \times \mathbb{T}^{n-k}$-action.

Remark. Even if this procedure yields examples of b-integrable systems on non-compact manifolds, we may consider Marsden-Weinstein reduction to obtain compact examples. Reduction in the b-setting is already described in [GMPS13] for abelian groups.

Chapter 7

KAM Theory for b-integrable systems

We have reviewed the basics of Hamiltonian equations and the classical KAM result in Section 2.6. For a b-function $H \in {}^bC^\infty(M)$ on a b-symplectic manifold, we can define the Hamiltonian equations in a completely analogous way: A function $g \in C^\infty(M)$ evolves according to the equation

$$\dot{g} = \{g, H\} = X_H(g). \tag{7.1}$$

Writing this equation in local coordinates, we obtain a set of equations similar to the classical Hamiltonian equations.

In this chapter we will extend the well-known KAM theorem for symplectic manifolds to the b-case, considering integrals and Hamiltonians that are b-functions.

The results of this chapter were published in [KMS] (joint work with Eva Miranda and Geoffrey Scott).

7.1 Equations of motion

The motion induced by any b-function H according to Equation (7.1) leaves the hypersurface invariant, since X_H is a Poisson vector field.

Assume that we are given a b-integrable system on M, which is compatible with the b-function H in the sense that H only depends on the action coordinates y,

$$H(\varphi, y) = k \log |y_1| + h(y).$$

This is a natural condition, since the physical interpretation of integrals is that of constants of motion.

From the action-angle coordinate theorem for b-symplectic manifolds, Theorem A, it follows that we can semilocally (around a Liouville torus) replace the given b-integrable system by the functions $y_1, \dots, y_n$ on $\mathbb{T}^n \times B^n$, where the y_i are the projections to the i-th component of B^n and the b-symplectic structure is

$$\frac{c}{y_1} d\varphi_1 \wedge dy_1 + \sum_{i=2}^{n-1} d\varphi_i \wedge dy_i. \tag{7.2}$$

Observe that y_1 is a local defining function for the critical hypersurface and that the latter is foliated into $(n-2)$-dimensional symplectic leaves given by the level sets of φ_1.

Computing the Hamiltonian vector field of H with respect to the b-symplectic structure (7.2), we arrive at the following equations of motion:

$$\begin{aligned}
\dot{\varphi}_1 &= \frac{k}{c} + \frac{y_1}{c} \frac{\partial h}{\partial y_1}(y) \\
\dot{\varphi}_i &= \frac{\partial h}{\partial y_i}(y) \qquad i = 2, \dots, n \\
\dot{y}_1 &= 0 \\
\dot{y}_i &= 0 \qquad i = 2, \dots, n.
\end{aligned}$$

We see that the motion is quasiperiodic. On Z the angle coordinates $(\varphi_1, \dots, \varphi_n)$ evolve with frequency

$$\left(\frac{k}{c}, \frac{\partial h}{\partial y_2}(y), \dots, \frac{\partial h}{\partial y_n}(y) \right) =: \left(\frac{k}{c}, \tilde{\omega} \right)$$

and the motion is constrained to an n-torus $\mathbb{T}^n \times \{\text{const}\} \subset Z$. In terms of notation, if $x \in \mathbb{R}^n$ is any n-vector, we will write $\tilde{x}$ for the $\mathbb{R}^{n-1}$ vector obtained by omitting the first component $\tilde{x} := (x_2, \dots, x_n)$.

7.1.1 Perturbed equations

We want to study the stability of the quasi-periodic motion inside Z, i.e. the behaviour of the system upon adding a perturbation $\epsilon P \in {}^bC^\infty(M)$ to the Hamiltonian. Thus the general form of the perturbation is

$$P(\varphi, y) = k' \log |y_1| + f(\varphi, y) \tag{7.3}$$

and the equations of motion of the perturbed system, given by the Hamiltonian $H + \epsilon P$, are:

$$\begin{aligned}
\dot{\varphi}_1 &= \frac{k + \epsilon k'}{c} + \frac{y_1}{c}\frac{\partial}{\partial y_1}\left(h(y) + \epsilon f(\varphi, y)\right) \\
\dot{\varphi}_i &= \frac{\partial}{\partial y_i}\left(g(y) + f(\varphi, y)\right) = \tilde{\omega}_i(y) + \epsilon\frac{\partial f}{\partial y_i}(\varphi, y), \qquad i = 2, \ldots, n \\
\dot{y}_1 &= -\epsilon\frac{y_1}{c}\frac{\partial f}{\partial \varphi_1}(\varphi, y) \\
\dot{y}_i &= -\epsilon\frac{\partial f}{\partial \varphi_i}(\varphi, y).
\end{aligned} \tag{7.4}$$

Notice that the hypersurface $Z = \{y_1 = 0\}$ is preserved by the perturbation.

In the following discussion about stability of the orbits we restrict ourselves to the case where the motion starts inside Z (and necessarily remains there); the other case is covered by the classical KAM theorem for symplectic manifolds.

We want to consider the case where the function f in the expression (7.3) for the perturbation P has the form

$$f(\varphi, y) = f_1(\tilde{\varphi}, y) + y_1 f_2(\varphi, y) + f_3(\varphi_1, y_1), \tag{7.5}$$

where f_1 is an analytic function and f_2, f_3 are smooth functions.

Remark. In particular, f has the above form if it does not depend on φ_1.

7.2 A KAM theorem for b-symplectic manifolds

We now show a stability result of KAM type for b-integrable Hamiltonian systems under perturbations by b-functions of the form described in Equation (7.5).

Theorem D (KAM Theorem for b-symplectic manifolds)**.** *Let $\mathbb{T}^n \times B_r^n$ be endowed with standard coordinates (φ, y) and the b-symplectic structure* (7.2)*. Consider a b-function*

$$H = k \log |y_1| + h(y)$$

on this manifold, where h is analytic. Let y_0 be a point in B_r^n with first component equal to zero, so that the corresponding level set $\mathbb{T}^n \times \{y_0\}$ lies inside the critical hypersurface Z.

Assume that the frequency map

$$\tilde{\omega} : B_r^n \to \mathbb{R}^{n-1}, \quad \tilde{\omega}(y) := \frac{\partial h}{\partial \tilde{y}}(y)$$

has a Diophantine value $\tilde{\omega} := \tilde{\omega}(y_0)$ *at* $y_0 \in B^n$ *and that it is non-degenerate at* y_0 *in the sense that the Jacobian* $\frac{\partial \tilde{\omega}}{\partial \tilde{y}}(y_0)$ *is regular.*

Then the torus $\mathbb{T}^n \times \{y_0\}$ *persists under sufficiently small perturbations of* H *which have the form mentioned above, i.e. they are given by* ϵP*, where* $\epsilon \in \mathbb{R}$ *and* $P \in^b C^\infty(\mathbb{T}^n \times B_r^n)$ *has the form*

$$\begin{aligned} P(\varphi, y) &= k' \log |y_1| + f(\varphi, y) \\ f(\varphi, y) &= f_1(\tilde{\varphi}, y) + y_1 f_2(\varphi, y) + f_3(\varphi_1, y_1). \end{aligned}$$

More precisely, if $|\epsilon|$ *is sufficiently small, then the perturbed system*

$$H_\epsilon = H + \epsilon P$$

admits an invariant torus $\mathcal{T}$*.*

Moreover, there exists a diffeomorphism $\mathbb{T}^n \to \mathcal{T}$ *close*[1] *to the identity taking the flow* γ^t *of the perturbed system on* $\mathcal{T}$ *to the linear flow on* $\mathbb{T}^n$ *with frequency vector*

$$\left(\frac{k + \epsilon k'}{c}, \tilde{\omega} \right).$$

Proof. First assume that $y_0 = 0$. We will prove the general case later on. As a purely formal step, we consider the restrictions of h and f_1 to Z as functions on B_r^{n-1} resp. $\mathbb{T}^{n-1} \times B_r^{n-1}$:

$$\overline{h}(\tilde{y}) := h(0, \tilde{y}), \qquad \overline{f}_1(\tilde{\varphi}, \tilde{y}) := f_1(\tilde{\varphi}, 0, \tilde{y}).$$

By the Kolmogorov theorem for symplectic manifolds, Theorem 2.6.4, there exists a constant $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ there is a symplectomorphism $\overline{\psi} : \mathbb{T}^{n-1} \times B_{r_*}^{n-1} \to \mathbb{T}^{n-1} \times B_r^{n-1}$ such that

$$(\overline{h} + \epsilon \overline{f}_1) \circ \overline{\psi} = h_*$$

is a function in Kolmogorov normal form on $\mathbb{T}^{n-1} \times B_{r_*}^{n-1}$ with frequency vector $\tilde{\omega}$. Denoting $\overline{\psi}(\tilde{\varphi}, \tilde{y}) =: (\tilde{\varphi}', \tilde{y}')$ we define a Poisson diffeomorphism

$$\psi : \mathbb{T}^n \times B_{r_*}^n \to \mathbb{T}^n \times B_r^n, \quad \psi(\varphi, y) := (\varphi_1, \tilde{\varphi}', y_1, \tilde{y}').$$

[1] By saying that the diffeomorphism is "ϵ-close to the identity" we mean that, for given H, P and r, there is a constant C such that $\|\psi - \mathrm{id}\| < C\epsilon$.

Then for $(\varphi, y) \in Z \subset \mathbb{T}^n \times B^n_{r_*}$:

$$\begin{aligned}(H{+}\epsilon P) \circ \psi(\varphi, y) &= (H + \epsilon P)(\varphi_1, \tilde{\varphi}', y_1, \tilde{y}') = \\ &= (k + \epsilon k') \log |y_1| + \epsilon y_1 f_2(\varphi_1, \tilde{\varphi}', y_1, \tilde{y}') + \epsilon f_3(\varphi_1, y_1) + \\ &\quad + h(y_1, \tilde{y}') + \epsilon f_1(\tilde{\varphi}', y_1, \tilde{y}').\end{aligned}$$

The first term of the above expression determines the motion of φ_1. The next two terms have no effect on the motion on Z. Let us consider the last two terms. Looking at the equations of motion (7.4), we see that the motion on Z does not depend on the partial derivative of the smooth part of the Hamiltonian with respect to y_1. In other words, we obtain the same motion on Z if we set $y_1 = 0$ in the smooth part of the Hamiltonian. Then the last two terms can be written as

$$h(0, \tilde{y}') + \epsilon f_1(\tilde{\varphi}', 0, \tilde{y}') = \overline{h}(\tilde{y}') + \epsilon \overline{f}(\tilde{\varphi}', \tilde{y}') = h_*(\tilde{\varphi}, \tilde{y}).$$

Recall that h_* is of Kolmogorov normal form with frequency vector $\tilde{\omega}$, i.e.

$$h_*(\tilde{\varphi}, \tilde{y}) = E_* + \tilde{\omega} \cdot \tilde{y} + Q_*(\tilde{\varphi}, \tilde{y}),$$

for some $E_* \in \mathbb{R}$, $Q_* = \mathcal{O}(|y|^2)$. Therefore, by looking at the equations of motion (7.4), we see that the trajectories of $H_* := (H + \epsilon P) \circ \psi$ on Z are precisely given by quasi-periodic motion with frequency $(\frac{k+\epsilon k'}{c}, \tilde{\omega})$ on the tori $\mathbb{T}^n \times \{m\}$ for $m \in B_{r_*}$.

Since ψ is a Poisson diffeomorphism, the flow of the Hamiltonian vector field associated to $H{+}\epsilon P$ is conjugated under ψ to the flow of the Hamiltonian vector field associated to H_*:

$$\gamma^t_{X_{H+\epsilon P}} = \psi \circ \gamma^t_{X_{H_*}} \circ \psi^{-1}.$$

The flow $\gamma^t_{X_{H_*}}$ leaves the torus $\mathbb{T}^n \times \{0\}$ invariant and therefore also the flow of the perturbed system $\gamma^t_{X_{H+\epsilon P}}$ leaves $\mathcal{T} := \psi(\mathbb{T}^n \times \{0\}) \cong \mathbb{T}^n$ invariant.

In conclusion, the motion induced by $H + \epsilon P$ on $\mathcal{T}$ is conjugated via ψ to the quasi-periodic motion on $\mathbb{T}^n \times \{0\}$ with frequency

$$\left(\frac{k + \epsilon k'}{c}, \tilde{\omega}\right).$$

Since the diffeomorphism $\overline{\psi}$ obtained from the Kolmogorov theorem for symplectic manifolds is ϵ-close to the identity, the transformation ψ we construct is also ϵ-close to the identity.

Now consider the case where $y_0 \neq 0$. Let τ be the translation which takes y_0 to 0. Note that τ only changes the last $n-1$ components since we assume that the first component of y_0 is already 0, so in particular τ is a Poisson diffeomorphism

$$\mathbb{T}^n \times B_{r'}(y_0) \to \mathbb{T}^n \times B_{r'}(0),$$

where $B_{r'}(y_0)$ is a ball around y_0 of some radius $r' > 0$ contained in the original ball B_r^n and we endow the sets with the b-symplectic structure inherited from $\mathbb{T}^n \times B_r^n$. Now we apply the argument above (case $y_0 = 0$) to the Hamiltonian $H \circ \tau$ and the perturbation $P \circ \tau$. Denote the diffeomorphism obtained there by ψ_0,

$$\psi_0 : \mathbb{T}^n \times B^n_{r'_*} \to \mathbb{T}^n \times B^n_{r'}.$$

Setting $\psi := \tau \circ \psi_0 \circ \tau^{-1}$ we obtain a Poisson diffeomorphism

$$\psi : \mathbb{T}^n \times B_{r'_*}(y_0) \to \mathbb{T}^n \times B^n_{r'}(y_0)$$

which is ϵ-close to the identity and conjugates the motion on

$$\mathcal{T} := \psi(\mathbb{T}^n \times \{y_0\})$$

to quasi-periodic motion on $\mathbb{T}^n \times \{y_0\}$ with frequency vector given by

$$\left(\frac{k + \epsilon k'}{c}, \tilde{\omega}\right).$$

□

Remark. Note that the first component of the frequency vector may change; this is the one that determines the "velocity" of the motion in the direction transverse to the symplectic leaves inside Z. Only the last $n-1$ components of ω are preserved. Moreover, since we only assume the Diophantine condition for the last $n-1$ components, the orbit through p_0 might not fill the whole torus $\mathbb{T}^n \times \{y_0\}$ densely. However, even in these cases the torus $\mathbb{T}^n \times \{y_0\}$ is invariant.

Remark. A special case is where the functions H and P are smooth, i.e. the log-component is zero:

$$H = h \in C^\infty(B_r^n), \qquad P = f \in C^\infty(\mathbb{T}^n \times B_r^n).$$

In this case we do not have to make the assumption that f has the form given in Equation (7.5) to obtain stability of the orbits inside Z. From the equations of motion it is clear that the trajectory starting at a point inside a symplectic leaf $\mathcal{L} \subset Z$ will stay inside the leaf. This is true also after adding the perturbation. Hence the stability of the orbit follows directly from the symplectic KAM theorem: If H is in b-Kolmogorov normal form with vanishing log component (i.e. a C^∞ function) and with Diophantine frequency vector $\tilde{\omega}$ and if P is any C^∞ perturbation, then there is a symplectomorphism on a neighborhood of the orbit *inside* $\mathcal{L}$ which is close to the identity and takes the perturbed orbit to a nearby $n-1$ torus $\{\varphi_1\} \times \mathbb{T}^{n-1} \times \{y\}$. The perturbed motion is conjugated to linear motion in the $\tilde{\varphi} := (\varphi_2, \ldots, \varphi_n)$ coordinates with frequency $\tilde{\omega}$. Note that we only transform inside the leaf $\mathcal{L}$ here – for showing stability this is sufficient.

Remark. In view of Example 4.2, this KAM theorem can be employed to study perturbations of integrable systems on manifolds with boundary.

Chapter 8

Singular symplectic structures in celestial mechanics

We have seen how to construct examples of b-integrable systems in Section 4.2, using the recipe of gluing together smooth integrable systems on manifolds with boundary which have the right asymptotics near the boundary. In this chapter we will discuss some natural examples arising in physics, where certain common transformations do not preserve the symplectic form but yield a singular structure which fails to be symplectic along a hypersurface. These singularities can have different forms, we will encounter b^m-symplectic structures and m-folded symplectic structures and sophistications of these types. We refer the reader to Section 2.3, where the definitions and main properties of these structures were introduced. The main results of this chapter were published in [DKM] (joint work with Amadeu Delshams and Eva Miranda).

8.1 Singular transformations in the Kepler problem

In this section we consider some classical transformations which are used to study the n-body problem and which are not symplectic. In the case $n = 2$ the system is integrable and an appropriate (non-canonical) transformation can be used to solve it. Other transformations are employed to study the dynamics of the n-body problem close to singularities, which emerge in the case of collisions or at the "line at infinity".

8.1.1 The two-body problem

We have already discussed the two-body problem in Section 3.1.1 and we have seen how the existence of integrals - in this case linear and angular momentum - reduces the problem to two degrees of freedom, the **planar Kepler problem**. This is the system of only one body with mass m moving on a plane under the influence of a central graviational potential. Explicitly, the Hamiltonian of this system, which has phase space $\mathbb{R}^2\backslash\{0\} \times \mathbb{R}^2$, is given by

$$H(w, W) = \frac{\|W\|^2}{2} - \frac{\mu}{\|w\|}, \qquad (w, W) \in (\mathbb{R}^2\backslash\{0\}) \times \mathbb{R}^2, \tag{8.1}$$

where $\mu := \mathcal{G}m$ is the so-called *gravitational parameter*. The corresponding second-order differential equation is

$$\ddot{w} = -\frac{\mu w}{\|w\|^3}, \qquad w \in \mathbb{R}^2. \tag{8.2}$$

8.1.2 Levi-Civita coordinates

We identify $\mathbb{R}^2$ with $\mathbb{C}$ and view Equation (8.2) as a differential equation over $\mathbb{C}$. In this way we can introduce a new coordinate u via

$$u^2 = w.$$

This step is called the **Levi-Civita regularization procedure** or **conformal squaring**. The components of u are called Levi-Civita coordinates.

Although we are interested in the *geometric* implications of this transformation (see Section 8.1.3), we want to explain the physical motivation behind it following [Wa]: We introduce a “fictious time” τ defined via the following equation:

$$dt = \frac{r}{c} d\tau, \quad r = \|w\|,$$

where c is a non-zero parameter. (For our purposes we can assume $c = 1$, which is called Sundman’s choice [Wa].) This means that when r is small, i.e. the distance between the two bodies is small, time intervals in the τ variable become larger (“slow-motion movie”).

We denote differentiation with respect to τ by a prime. Then, expressing the equation of the planar Kepler problem, Equation (8.2), in Levi-Civita coordinates u and the new time τ, a straightforward computation carried out in [Wa] yields the following equation:

$$2ru'' + \left(\frac{mu}{c^2} - 2\|u'\|^2\right) u = 0. \tag{8.3}$$

We now fix a value of the energy H,

$$H(w, W) = \text{const.} =: h.$$

Inserting $W = \dot{w}$ into the Hamiltonian (8.1) we obtain

$$\frac{\|\dot{w}\|^2}{2} - \frac{\mu}{\|w\|} = h.$$

A short calculation, expressing the left-hand side of this equation in the new coordinates u and new time derivatives $d/d\tau$, and multiplying both sides by r yields

$$2c^2\|u'\|^2 - \mu = rh.$$

In this way we can eliminate the non-linear term $\|u'\|^2$ in Equation (8.3):

$$2c^2u'' + hu = 0.$$

This is the well-known equation of the harmonic oscillator.

8.1.3 Geometrical structure

We want to study how the symplectic form changes under the Levi-Civita regularization. (The change of time is not relevant in this respect.)

Recall that the phase space coordinates are $(w, W) \in (\mathbb{R}^2 \backslash \{0\}) \times \mathbb{R}^2$ endowed with the canonical symplectic form. We leave the momentum coordinates W unchanged and express the space coordinates w in terms of u:

$$w_1 + iw_2 = w = u^2 = (u_1 + iu_2)^2 = u_1^2 - u_2^2 + 2iu_1u_2.$$

The real resp. imaginary part correspond to w_1 resp. w_2. Hence their differentials are

$$dw_1 = 2u_1du_1 - 2u_2du_2, \qquad dw_2 = 2u_1du_2 + 2u_2du_1$$

and the symplectic form is

$$\begin{aligned}\omega = dw_1 \wedge dW_1 + dw_2 \wedge dW_2 = {} & 2u_1du_1 \wedge dW_1 - 2u_2du_2 \wedge dW_1 \\ & + 2u_1du_2 \wedge dW_2 + 2u_2du_1 \wedge dW_2.\end{aligned}$$

We compute the wedge

$$\omega \wedge \omega = 4(u_1^2 - u_2^2)du_1 \wedge dW_1 \wedge du_2 \wedge dW_2.$$

We see that $\omega \wedge \omega$ does not cut the zero section of Λ^2T^*M transversally, since 0 is not a regular value of the function $u_1^2 - u_2^2$. Therefore the condition of being a folded symplectic structure is not satisfied; instead the singularity is non-degenerate of *hyperbolic* type.

Remark. A different version of the Levi-Civita transformation described in [Bo, Ce03] does not keep the momenta unchanged but transforms them in such a way that the total change is *symplectic*. More precisely the change is given by:

$$w = \frac{u^2}{2}, \qquad W = \frac{U}{\overline{u}},$$

where (u, U) are the new coordinates. In this case the symplectic form remains the standard one but the equations become more involved.

8.1.4 The Kepler problem in three dimensions - KS transformation

We now consider the Kepler problem in three dimensions and do not perform the step of reducing it to a planar system, i.e. we consider the second-order differential equation that we already encountered in the planar problem in three dimensions:

$$\ddot{w} = -\frac{\mu w}{\|w\|^3}, \qquad w \in \mathbb{R}^3. \tag{8.4}$$

This point of view is relevant e.g. for studying binary collisions in the three-body problem, which cannot be restricted to two dimensions in general.

Instead of working with complex numbers, the regularization procedure now employs the **quaternion algebra** $\mathbb{U}$, see [Wa]. Recall that $\mathbb{U}$ consists of objects of the form

$$u = u_0 + iu_1 + ju_2 + ku_3$$

where i, j, k are the three independent "imaginary" units and multiplication is defined via the non-commutative laws

$$ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

We identify the quaternion u with the vector $u = (u_0, u_1, u_2, u_3) \in \mathbb{R}^4$. Moreover, we introduce the star conjugation

$$u^* := u_0 + iu_1 + ju_2 - ku_3$$

Now instead of considering the squaring of complex numbers as in the previous section, we define the mapping

$$u \mapsto w := \frac{uu^*}{2}. \tag{8.5}$$

The image is the set of quaternions with vanishing k component and can be identified with $\mathbb{R}^3$. The preimage of a number $w = vv^*$ is given by the one-parameter family of quaternions of the form $v \cdot e^{k\theta} := v \cdot (\cos\theta + k \sin\theta)$ where $\theta \in S^1$.

Writing the transformation (8.5) explicitly (where we identify the image with $\mathbb{R}^3$), we have

$$\begin{aligned} w_0 &= \frac{u_0^2 - u_1^2 - u_2^2 + u_3^2}{2} \\ w_1 &= u_0u_1 - u_2u_3 \\ w_2 &= u_0u_2 + u_1u_3. \end{aligned}$$

This is known as the **Kustaanheimo-Stiefel (KS)** transformation. We choose a solution with vanishing k-component, i.e. $u_3 = 0$. Then the new space coordinates are (u_0, u_1, u_2).

Solving the Kepler problem in three dimensions. The KS transformation is used to simplify the equation of motion, similar to the approach used in the two-dimensional case. A new time variable τ is introduced and writing Equation (8.4) in terms of the new time coordinate and the space coordinates of the KS transform, a computation carried out in [Wa] shows that we obtain again the equation of the harmonic oscillator:

$$2u'' + hu = 0$$

where h again is the energy of the solution.

Geometric structure. Denoting the original space coordinates by (w_0, w_1, w_2) with conjugate momenta (W_0, W_1, W_2), the symplectic form becomes in the new space coordinates $(u_0, u_1, u_2) \in \mathbb{R}^3$ after the KS transform:

$$\begin{aligned} \omega =& dw_0 \wedge dW_0 + dw_1 \wedge dW_1 + dw_2 \wedge dW_2 = \\ =& (u_0du_0 - u_1du_1 - u_2du_2) \wedge dW_0 + (u_0du_1 + u_1du_0) \wedge dW_1 \\ &+ (u_0du_2 + u_2du_0) \wedge dW_2. \end{aligned}$$

In order to determine which kind of geometric structure this defines, we compute

$$\omega \wedge \omega \wedge \omega = (u_0^3 - u_1^2u_0 - u_2^2u_0)du_0 \wedge dW_0 \wedge du_1 \wedge dW_1 \wedge du_2 \wedge dW_2.$$

Observe that the coefficient of $\omega \wedge \omega \wedge \omega$ is $2u_0w_0$. This is a sophistication of m-folded symplectic structures.

8.2 Triple collisions in the three-body problem

In this section we focus on triple collisions for the three-body problem but the computation holds *mutatis mutandis* for the total collapse in the n-body problem i.e. collision of n bodies.

Following [Mo1980], [Mo1996] and [Mc], consider the system of three bodies with masses m_1, m_2, m_3 and positions

$$\mathbf{q_1} = (q_1, q_2, q_3), \mathbf{q_2} = (q_4, q_5, q_6), \mathbf{q_3} = (q_7, q_8, q_9) \in \mathbb{R}^3.$$

Similarly we denote the momenta of the bodies by

$$\mathbf{p_1} = (p_1, p_2, p_3), \mathbf{p_2} = (p_4, p_5, p_6), \mathbf{p_3} = (p_7, p_8, p_9) \in \mathbb{R}^3.$$

Moreover, we set $q = (q_1, \ldots, q_9)$ and $p = (p_1, \ldots, p_9) \in \mathbb{R}^9$. We define the 9×9 matrix $M := \operatorname{diag}(m_1, m_1, m_1, m_2, m_2, m_2, m_3, m_3, m_3)$.

Recall that the three-body problem has Hamiltonian

$$H(q, p) = \frac{p^T M^{-1} p}{2} - U(q)$$

where the potential U is given by

$$U(q) = \frac{m_1 m_2}{|\mathbf{q}_1 - \mathbf{q}_2|} + \frac{m_2 m_3}{|\mathbf{q}_2 - \mathbf{q}_3|} + \frac{m_1 m_3}{|\mathbf{q}_1 - \mathbf{q}_3|}.$$

We assume that we are working in central coordinates, so the centre of mass remains at the origin:

$$m_1 \mathbf{q_1} + m_2 \mathbf{q_2} + m_3 \mathbf{q_3} = 0.$$

McGehee transformation. In order to study the qualitative behaviour of orbits near the triple collision singularity, we introduce the following change of coordinates, which was first proposed by McGehee [Mc]:

$$r := \sqrt{q^T M q}, \qquad s := \frac{q}{r}, \qquad z := p\sqrt{r}. \tag{8.6}$$

Essentially, these are spherical coordinates since s lies on the unit-sphere in $\mathbb{R}^9$ with respect to the metric given by M,

$$s^T M s = 1.$$

Note that $r = 0$ corresponds to triple collisions.

Following [Mo1980], we compute the equations of motion in the new coordinates and with respect to a new time τ,

$$d\tau = r^{\frac{2}{3}} dt.$$

We denote differentiation with respect to τ by a prime. We introduce the function

$$v := s \cdot z,$$

where the dot denotes the standard inner product on $\mathbb{R}^9$.

Then the equations of motion are

$$\begin{aligned} r' &= vr, \\ s' &= M^{-1}z - vs, \\ z' &= \nabla U(s) + \frac{vz}{2}. \end{aligned} \tag{8.7}$$

The energy and angular momentum are preserved quantities and we fix a value $H(q,p) = h$ resp. $p \times q = \omega$, resulting in the following equations:

$$\frac{z^T M^{-1} z}{2} - U(s) = rh, \quad z \times s = r^{\frac{1}{2}}\omega.$$

We have $v' = s' \cdot z + s \cdot z'$ and inserting the equations of motion (8.7) as well as the conservation of energy we obtain the so-called **Lagrange equation**:

$$v' + \frac{v^2}{2} = \frac{z^T M^{-1} z}{2} + rh.$$

We denote by bold letters the $\mathbb{R}^3$ vectors $\mathbf{s}_1 = (s_1, s_2, s_3)$, $\mathbf{s}_2 = (s_4, s_5, s_6)$ etc.

Consider the inequality

$$\sum_{j=1,2,3} |\mathbf{s}_j|^2 |\mathbf{z}_j|^2 = \sum_{j=1,2,3} |\mathbf{s}_j|^2 m_j m_j^{-1} |\mathbf{z}_j|^2 \leq \sum_{j=1,2,3} |\mathbf{s}_j|^2 m_j \sum_{j=1,2,3} |\mathbf{z}_j|^2 m_j^{-1}.$$

Since we have $s^T M s = 1$ the right hand side of this inequality is equal to $z^T M^{-1} z$. Inserting

$$|\mathbf{s}_j|^2 |\mathbf{z}_j|^2 = (\mathbf{s}_j \cdot \mathbf{z}_j)^2 + |\mathbf{s}_j \times \mathbf{z}_j|^2$$

the above inequality becomes

$$z^T M^{-1} z \geq (\underbrace{s \cdot z}_{=v})^2 + |\underbrace{s \times z}_{=\omega}|^2.$$

Using Lagrange's equation, we obtain a lower bound for v':

$$v' \geq rh + \frac{r^{-1}|\omega|^2}{2}.$$

This relation is called **Sundman's inequality**. From the equations of motion (8.7) for r' we know that $v = (\ln r)'$ and therefore Sundman's inequality can be used to make assertions about the evolution of r, i.e. the distance from triple collision. More precisely, in the case of non-zero angular momentum $|\omega| \neq 0$, we can deduce the following information about the qualitative behaviour near triple collisions: Assume that $r^2 \leq \frac{|\omega^2|}{4|h|}$. Then

$$v' \geq \frac{|\omega|^2}{4r}.$$

Therefore, if from a certain time t_0 on, r is smaller than a fixed number $r_0 > 0$, the derivative of v has a positive lower bound and therefore $v = (\ln r)'$ will eventually become positive, meaning that r is increasing. Therefore, r can never become zero, in other words triple collisions are impossible.

Remark. The case of zero angular momentum $\omega = 0$ is different and triple collisions become possible. We refer to [Mo1980] for a detailed discussion.

Geometric structure. Based on Equation (8.6) we want to introduce a well-defined chart and study the geometric structure that this change of coordinates entails.

We restrict to the subset $q_9 > 0$ of the phase space $\mathbb{R}^9 \times \mathbb{R}^9$ and consider the coordinates

$$(r, s_1, \ldots, s_8, z_1, \ldots, z_9)$$

where r, s, z are defined in Equation (8.6). The inverse of this chart is

$$\begin{aligned} q_i &= rs_i, \qquad i = 1, \ldots, 8 \\ q_9 &= r\sqrt{\frac{1 - \sum_{i=1}^{8} s_i^2 m_i}{m_9}}, \\ p_i &= \frac{z_i}{\sqrt{r}}, \qquad i = 1, \ldots, 9. \end{aligned}$$

Computing the differentials one sees that the standard symplectic form $\sum_{i=1}^{9} dq_i \wedge$

dp_i becomes

$$\sum_{i=1}^{8}\left(\frac{s_i}{\sqrt{r}}dr\wedge dz_i+\sqrt{r}ds_i\wedge dz_i-\frac{z_i}{2\sqrt{r}}ds_i\wedge dr\right)+$$
$$+\frac{1}{\sqrt{m_9 r\mu}}\left(\mu dr\wedge dz_9-r\sum_{i=1}^{8}m_i s_i ds_i\wedge dz_9+\frac{z_9}{2}\sum_{i=1}^{8}m_i s_i ds_i\wedge dr\right),$$

where we have introduced the function $\mu := 1-\sum_{i=1}^{8}s_i^2 m_i$ to simplify notation.

We compute the top wedge of the structure:

$$\Lambda_{i=1}^{9}dq_i\wedge dp_i=\sqrt{\frac{\mu r^7}{m_9}}ds_1\wedge dz_1\wedge ds_2\wedge dz_2\wedge\ldots\wedge ds_8\wedge dz_8\wedge dr\wedge dz_9,$$

hence for $r=0$ this expression vanishes to order $\frac{7}{2}$ and is a $\frac{7}{2}$-folded symplectic structure.

8.3 The elliptic restricted three-body problem: McGehee coordinates

Let us now consider a special case of the three-body problem where one of the bodies is assumed to have negligible mass. E.g. this happens if the system is given by Sun, Jupiter and an asteroid. Then a useful approximation is to assume that the motion of the two heavy bodies, called primaries (here Sun and Jupiter), is independent of the small body, hence given by Kepler's law for the two-body problem. This problem is known as the **restricted** three-body problem. We will moreover assume that all the three bodies move in a plane (*planar restricted three-body problem*).

We are interested in the resulting dynamical system for the small body (the asteroid), which moves under the influence of the time-dependent gravitational potential of the primaries

$$U(q,t)=\frac{1-\mu}{|q-q_1|}+\frac{\mu}{|q-q_2|},$$

where we assume that the masses of the primaries are normalized and given by μ resp. $1-\mu$; their time-dependent positions are $q_1=q_1(t)$ resp. $q_2=q_2(t)$.

The Hamiltonian of the system is given by

$$H(q, p, t) = \frac{p^2}{2} - U(q, t), \quad (q, p) \in \mathbb{R}^2 \times \mathbb{R}^2,$$

where $p = \dot{q}$ is the momentum of the asteroid. The equations of motion are then obtained in the usual way,

$$\ddot{q}(t) = p(t) = -\frac{\partial H}{\partial q} = (1-\mu)\frac{q_1 - q}{|q_1 - q|^3} + \mu \frac{q_2 - q}{|q_2 - q|^3}$$

In the *elliptic* restricted three-body problem, the primaries are assumed to move around their center of mass on ellipses. Following [DKRS], we introduce polar coordinates to describe the motion of the small body. Then for $q = (X, Y) \in \mathbb{R}^2 \backslash \{0\}$, we have

$$X = \rho \cos \alpha, Y = \rho \sin \alpha, \qquad (\rho, \alpha) \in \mathbb{R}^+ \times \mathbb{T}$$

The momenta $p = (P_X, P_Y)$ are transformed in such a way that the total change of coordinates

$$(X, Y, P_X, P_Y) \mapsto (\rho, \alpha, P_\rho =: y, P_\alpha =: G)$$

is canonical, i.e. the symplectic structure remains the same. The primaries move according to the following relations:

$$q_1 = \mu r(\cos f, \sin f), \quad q_2 = -(1-\mu) r(\cos f, \sin f)$$

where r is the distance between the two primaries and depends on the eccentricity e and the *true anomaly* f,

$$r = \frac{1 - e^2}{1 + e \cos f}$$

where f is given by the following differential equation

$$\dot{f} = \frac{(1 + e \cos f)^2}{(1 - e^2)^{3/2}}.$$

The Hamiltonian in these polar coordinates is given by

$$H^*(\rho, \alpha, y, G, t) = \frac{y^2}{2} + \frac{G^2}{2\rho} - U^*(\rho, \alpha, t)$$

with

$$U^*(\rho, \alpha, t) = U(\rho \cos \alpha, \rho \sin \alpha, t).$$

McGehee coordinates. To study the behaviour at $r = \infty$, a common procedure is to introduce the so-called *McGehee coordinates* (x, α, y, G), where

$$\rho = \frac{2}{x^2}, \quad x \in \mathbb{R}^+.$$

Then $\rho = \infty$ corresponds to the origin $x = 0$.

The Hamiltonian in McGehee coordinates is given by

$$H_*(x, \alpha, y, G, t) = \frac{y^2}{2} + \frac{x^4 G^2}{8} - U_*(x, \alpha, t)$$

where

$$U_*(x, \alpha, t) = U^* \left(\frac{2}{x^2}, \alpha, t \right).$$

In particular, for $\mu = 0$, the Hamiltonian becomes quadratic and time-independent (autonomous):

$$H_*(x, \alpha, y, G, t) = \frac{y^2}{2} + \frac{x^4 G^2}{8} - \frac{x^2}{2} \qquad \text{for } \mu = 0.$$

As we discuss below, the symplectic structure is not preserved under this change and therefore, the Hamiltonian equations of motion have to be computed with respect to a different structure.

The McGehee change of coordinates is employed in [DKRS] to study the so-called infinity manifold. The case $\mu > 0$ is viewed as a perturbation of the limit case $\mu = 0$, for which we have seen that the Hamiltonian has a very simple structure. The main result in [DKRS] is the existence of diffusive trajectories, i.e. trajectories with a "large" variation of angular momentum.

Geometric structure. The McGehee transformation is non-canonical i.e. the symplectic structure changes. Inserting the expression for ρ into the canonical symplectic form $d\rho \wedge dy + d\alpha \wedge dG$ shows that it is given by

$$-\frac{4}{x^3} dx \wedge dy + d\alpha \wedge dG, \qquad x \in \mathbb{R}^+.$$

This extends naturally to a b^3-symplectic structure on $\mathbb{R} \times \mathbb{T} \times \mathbb{R}^2$ in the sense of [Sc]. In particular, we immediately see that the hypersurface $\{x = 0\}$ is invariant.

Equivalently, the Poisson bracket is

$$\{f, g\} = -\frac{x^3}{4} \left(\frac{\partial f}{\partial g} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} \right) + \frac{\partial f}{\partial \alpha} \frac{\partial g}{\partial G} - \frac{\partial f}{\partial G} \frac{\partial g}{\partial \alpha}.$$

Such a Poisson structure is used extensively in [DKRS] to describe the dynamics close to the infinity manifold $x = y = 0$.

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